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By-Kirwin, Gerald J.

Inductors and Inductance-Resistance Networks.

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This programed booklet presents ideas related to inductors and inductance--resistance networks. It is designed for the engineering student who is familiar with differential equations and electrical networks. A variety of cases are considered with the idea of developing in the student a broad acquaintance with the inductor response. The booklet is divided into the following parts--(1) Real and Ideal Inductors, (2) A Limit to Inductor Current Change, (3) Changing Current and Voltage in Inductors, (4) Energy Storage in the Inductor, (5) Solving for the Current, and (6) Review. (RP)

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INDUCTORS AND INDUCTANCE-RESISTANCE NETWORKS

by

Gerald J. Kirwin

Electrical Engineering Department
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TABLE OF CONTENTS

	Topic	Pages
1.	Real and Ideal Inductors Basic i-v Relations in Ideal Inductor	1-19
2.	A Limit to Inductor Current Change	20-38
3.	Changing Current and Voltage in Inductors	39-74
4.	Energy Storage in the Inductor	75-94
5.	Solving for the Current	95-124
6.	Review	125-149

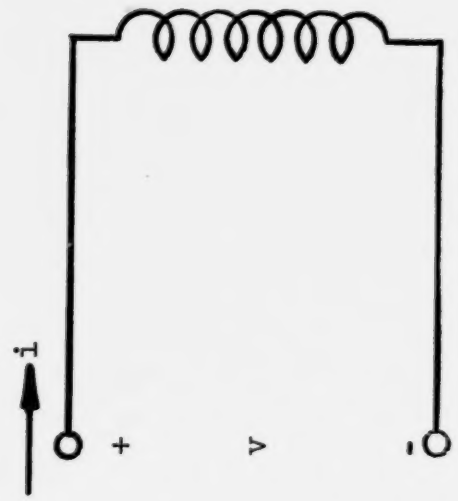


Fig. 1

Part I

Consider a length of wire, arrang[illegible]ed in the form of a coil, Fig. 1. An electric current, i, will be accompani[illegible]ed by a magnetic field which links the turns of the coil. The magnetic field will contribute to the terminal voltage, v, the contribution being proportional to the derivative of the current, $\frac{di}{dt}$. In addition, the resistance of the wir[illegible]e, R, will also contribute to the voltage. Write the equation for the voltage in terms of the above two components:

$$v = _____ + _____$$

$$v = L \frac{di}{dt} + Ri \qquad (1)$$

L - inductance

R - resistance

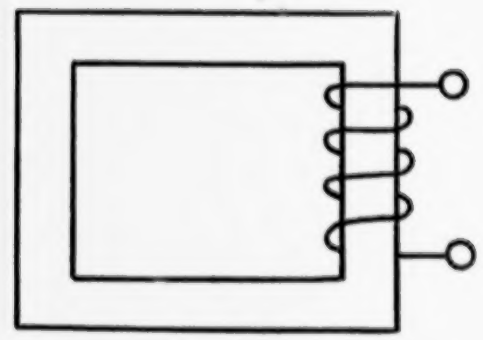


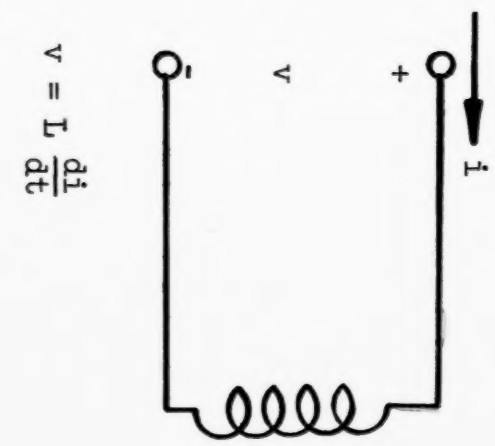
Fig. 2

In Eq. 1, the L di/dt term, called the inductive voltage, is a consequence of the changing magnetic field. In practice, this part of the terminal voltage can be very large by comparison with the resistive voltage, Ri.

One way to make L large is to wind the wire on a piece of soft iron, as in Fig. 2.

The smaller the ratio of the resistive voltage to the inductive voltage, the more nearly will the coil of wire act as an "ideal"inductor (in which the resistance is assumed to be zero).

Write the voltage-current relationship for an ideal inductor. Draw its circuit symbol, including voltage and current references.



$$v = L \frac{di}{dt}$$

(2)

One feature of the ideal inductor that needs emphasis is the fact that its inductance L is <u>constant</u>, independent of either <u>i</u> or di/dt. For the current and voltage references that have been chosen, L is a positive number. (Note that the convention shown on page 4 is similar to that used when putting references on a resistor.)

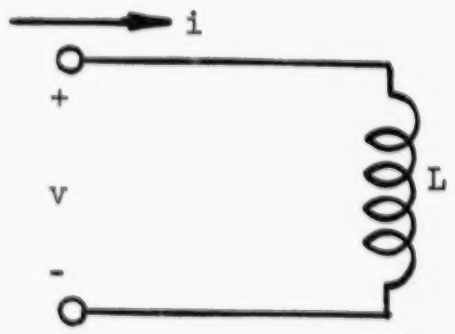
It will be helpful for you to formulate a definition of an ideal inductor in your own words.

An ideal inductor is _____

_____

_____.

An ideal inductor is a two-terminal device, symbolized below. It has a voltage-current relation, $v = L \frac{di}{dt}$. L is called the inductance and is a positive constant that is independent of the current.



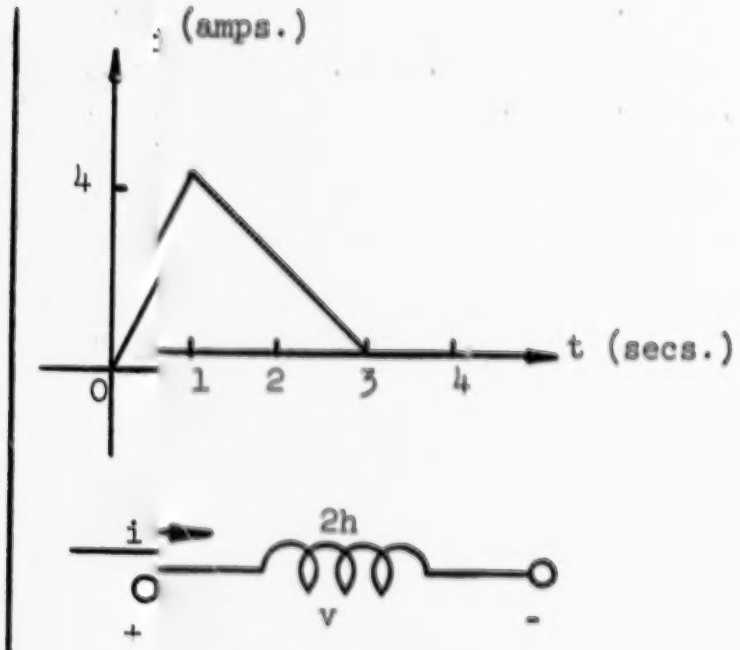


Fig. 3

In order to lead up to the study of networks containing inductors and resistors, we shall begin by considering the response of the inductor to specified currents or voltages. A variety of cases will be taken up with the plan of developing a broad acquaintance with the inductor response.

Suppose the current in a 2-henry inductor varies with time as indicated in Fig. 3. Construct a labeled sketch of the voltage.

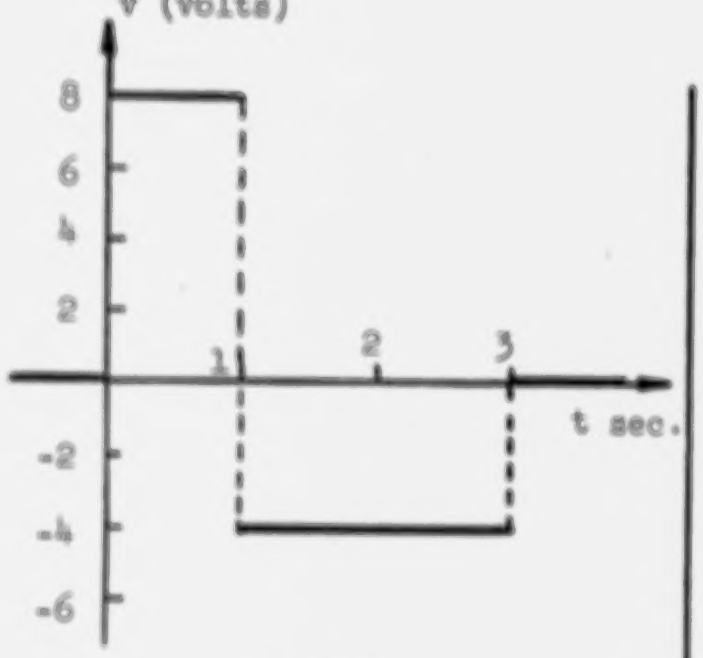

v (volts)
8
6
4
2
-2
-4
-6
1
2
3
t sec.

Fig. 4

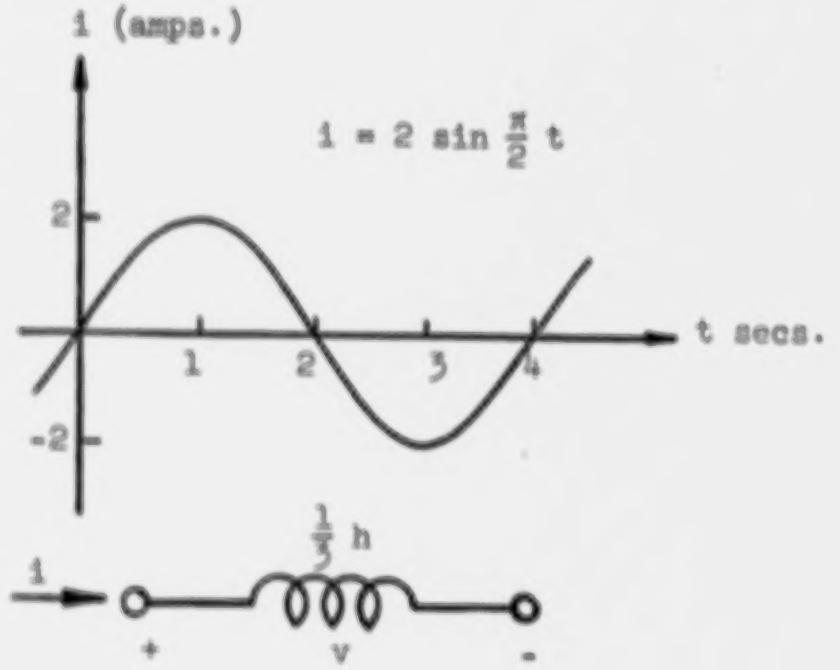
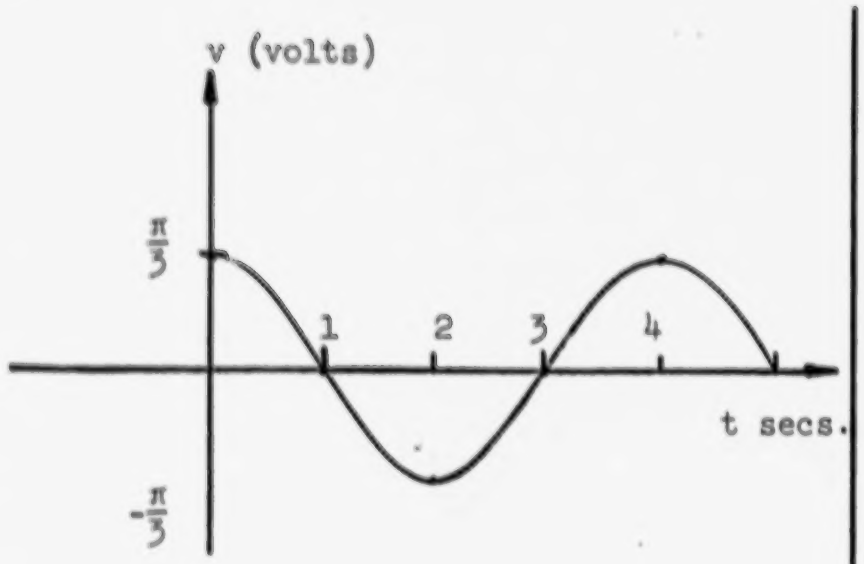

i (amps.)
i = 2 sin $\frac{\pi}{2}$ t
2
-2
1
2
3
4
t secs.
$\frac{1}{3}$ h
i
+
v
-

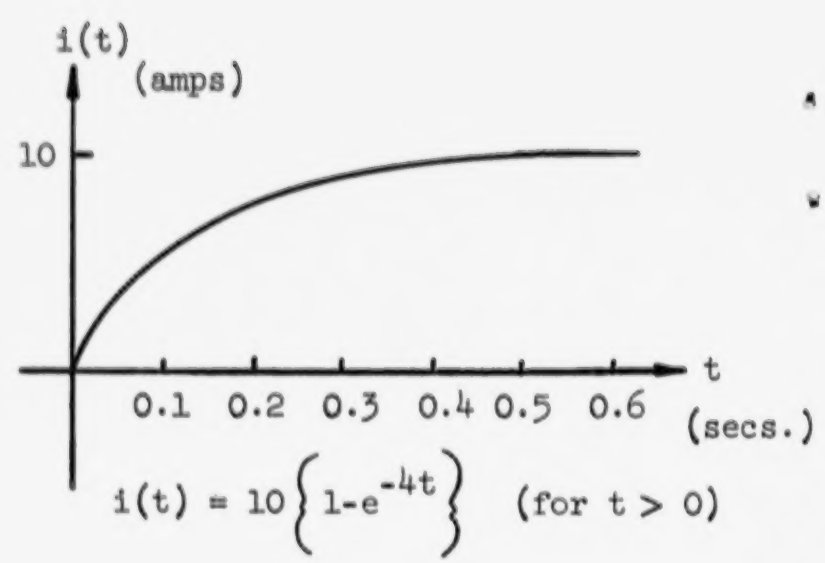
Fig. 5

In Fig. 4, notice that the inductor voltage may have jumps (discontinuities). Later, we shall learn that a similar statement <u>cannot</u> be made about the inductor current.

Consider the current depicted in Fig. 5. Determine the equation of the voltage across a $\frac{1}{5}$ henry inductor produced by this current. Also construct a labeled sketch of the voltage.



$$v = \frac{\pi}{3} \cos \frac{\pi}{2} t$$



$$i(t) = 10\left\{1-e^{-4t}\right\} \quad (\text{for } t > 0)$$

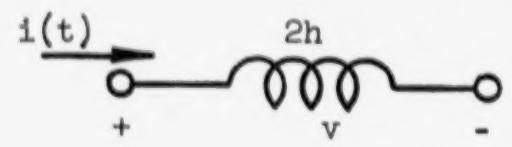
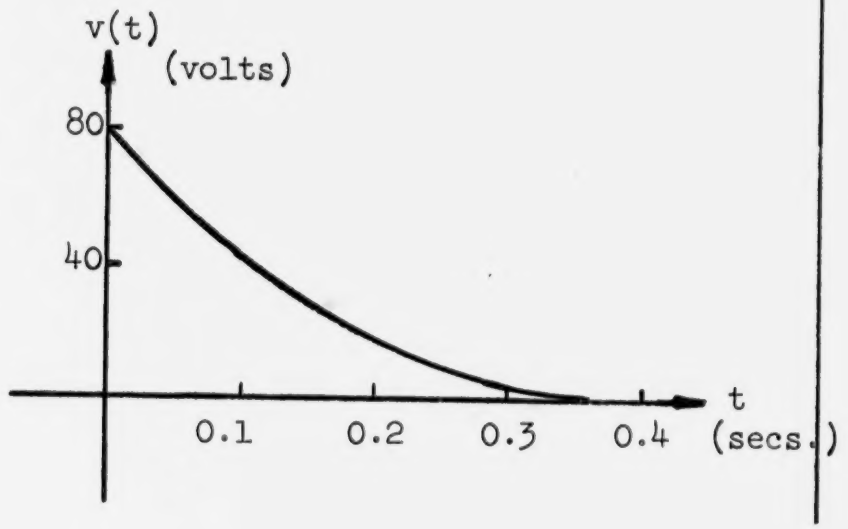


Fig. 6

The current in a certain two-henry inductor is given in Fig. 6. Determine the voltage as a function of time and sketch this function.

$$v(t) = L \frac{d}{dt} i(t)$$

$$v(t) = 80e^{-4t}$$



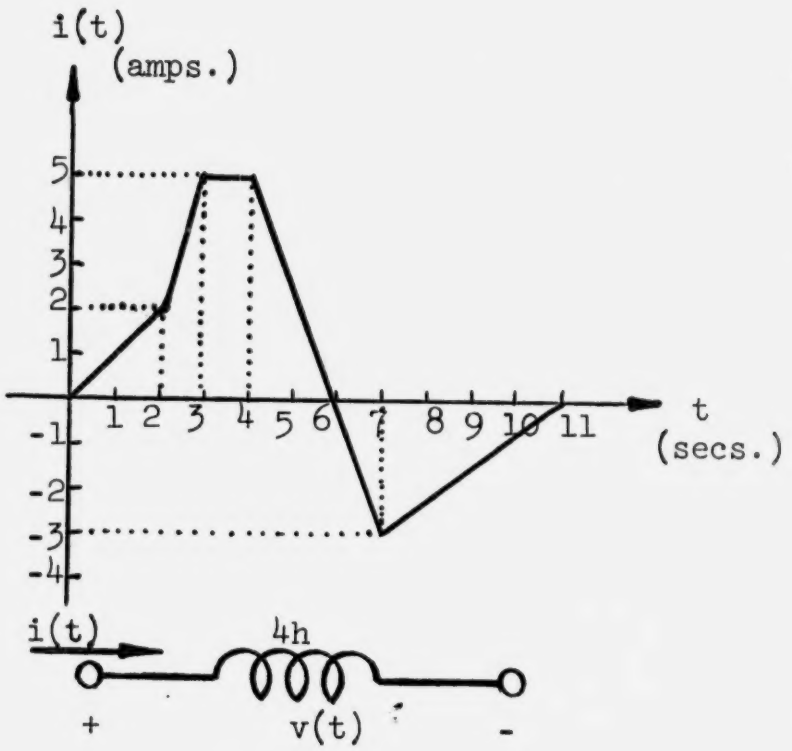
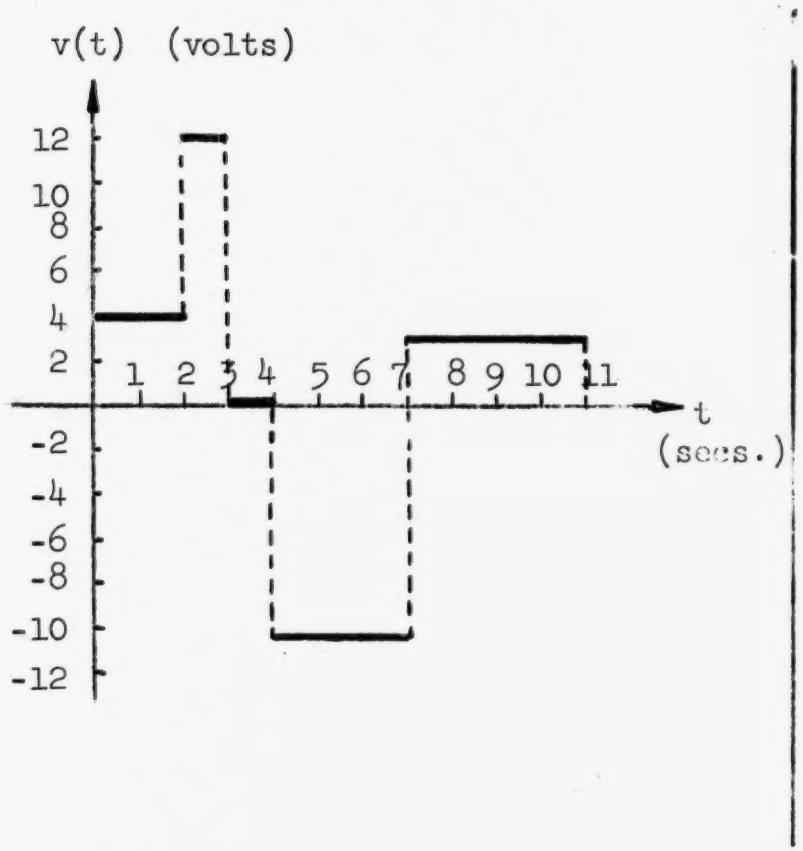


Fig. 7

Consider the current depicted in Fig. 7. Determine the voltage across a 4-henry inductor if the inductor carries this current. (You should be able to solve this problem by inspection.)


v(t) (volts)
12
10
8
6
4
2
-2
-4
-6
-8
-10
-12
1 2 3 4 5 6 7 8 9 10 11
t
(secs.)

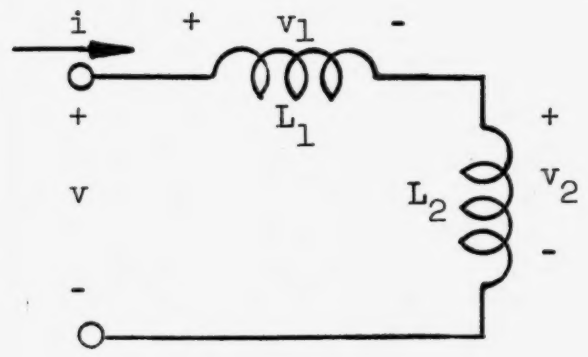

i
+
v1
-
+
L1
+
v
L2
v2
-
-

Fig. 8a

In Fig. 8a, two inductors are shown connected in series (which means that they carry the same current).

How is the total voltage v related to the voltage across each inductor?

v = _____

$$v = v_1 + v_2$$

$$= L_1 \frac{di}{dt} + L_2 \frac{di}{dt}$$

$$= (L_1 + L_2) \frac{di}{dt}$$

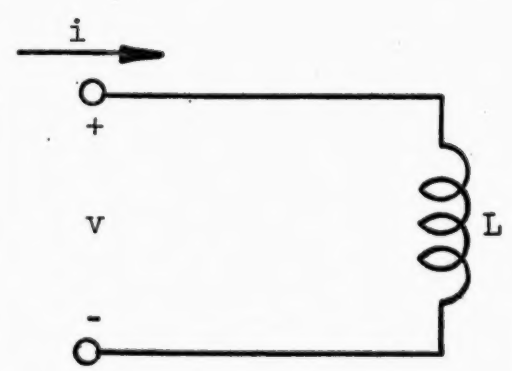


Fig. 8b

In Fig. 8b, a single inductor is indicated. If this single inductor is equivalent to the series connection of the two inductors in Fig. 8a, then each arrangement must have the same voltage-current relationship. Thus the two equations:

$$v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} \qquad \text{and} \qquad v = L \frac{di}{dt}$$

must express the same relationship between $\underline{v}$ and $\underline{i}$.

How must L_1, L_2, and L be interrelated to satisfy this requirement?

$L = L_1 + L_2$

Stated briefly in words:

"Inductors in series add".

Summary

Complete the following:

1) Doubling the time rate of change of the current in an inductor will _____ the voltage across the inductor.
2) In a certain inductor a current derivative of 4 amperes per millisecond causes a voltage of 200 volts. The inductance is L = _____ henry.
3) At a certain instant, the voltage across a 0.10-henry inductor is 100 volts. The rate of change of current is _____ amperes/second.
4) The voltage across a 2-henry inductor whose current is a steady 10 amperes equals _____ volts.
5) Draw and label (with references) the symbol for an ideal inductor. Write an equation for its v-i relationship.

Summary

1,4,5) You know these.

2) .050 henries

3) 1000 amperes/sec.

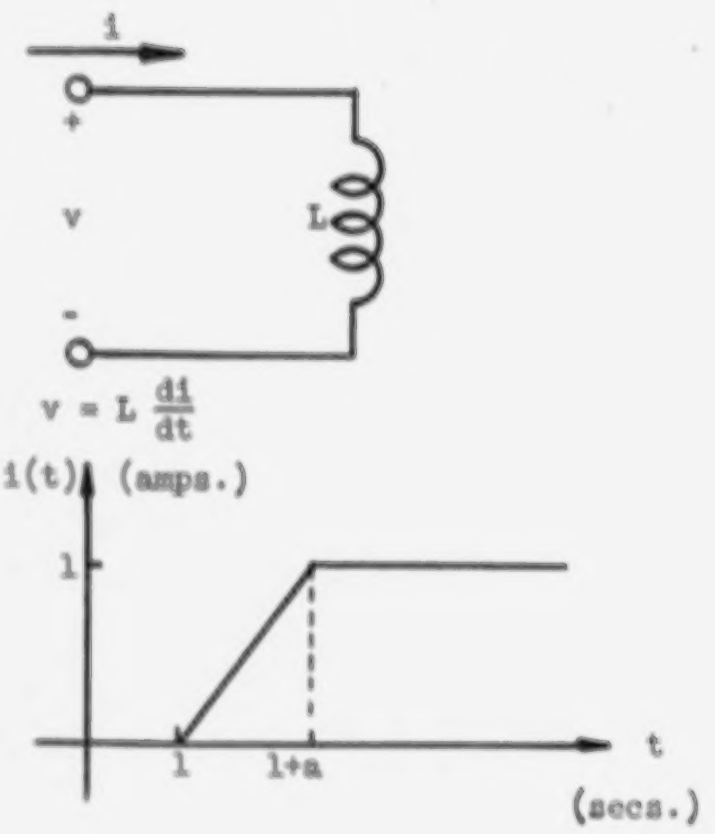


Fig. 9

Part II

A limit to inductor current change

Because inductor voltage is proportional to the rate of change current, the effect of a discontinuity, or sudden change in the current, needs to be considered. We can approach this problem by studying the current depicted in Fig. 9. At first we let a be a small but finite number. Then by letting a tend to zero, the current function will become discontinuous, and the effect upon the voltage may be observed.

Assume that a in Fig. 9 is a small, finite quantity and construct a scaled sketch of the voltage v(t) across an L henry inductor if the current is that ofFig. 9.

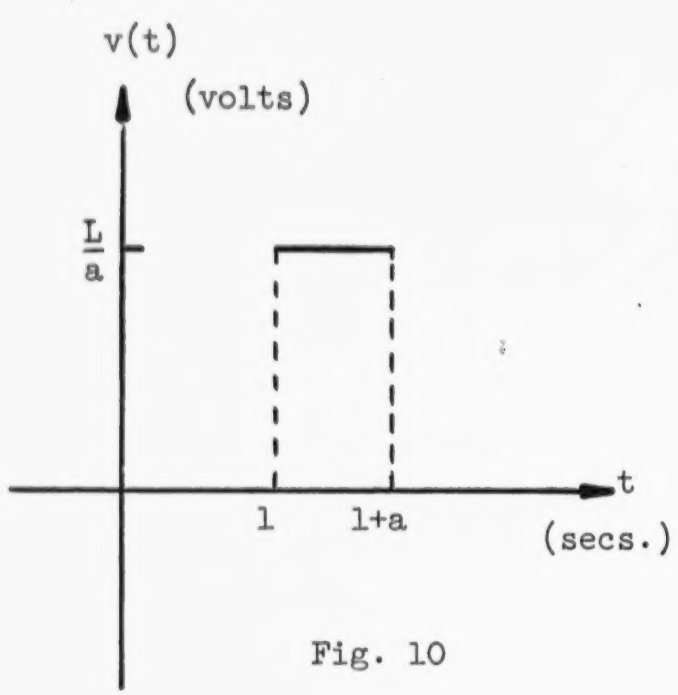


Fig. 10

From Fig. 10 it is clear that the amplitude of the voltage pulse (the peak voltage resulting from the change in current) varies inversely with a. Consequently, when a is small, the peak voltage produced is large. This is in keeping with the fact that a small value of a means a large value of di/dt.

How does the area of the voltage pulse (Fig. 10) depend upon a?

The area of the pulse does not depend upon a. This area is L volt seconds.

Later on, it will be shown that the area under the voltage pulse depends only on the inductance and the net change in the current.

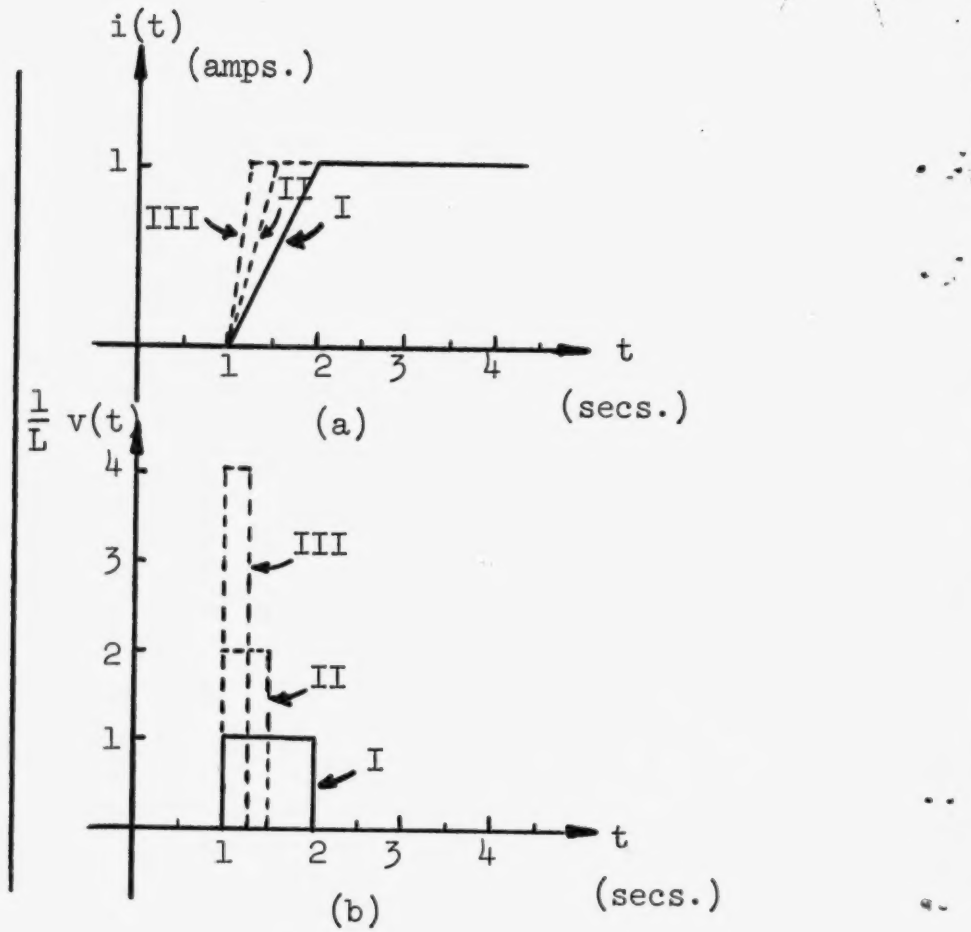


Fig. 11

In Fig. 11a the inductor current for each of three values of a is shown. The effect of reducing a on the voltage across the inductor is indicated in Fig. 11b, where the v(t)/L is plotted.

Let L = 3 henries and calculate the amplitude of the voltage pulse for each of the three values a depicted in Fig. 11.

Case I, v_{max} = _____ volts

Case II, v_{max} = _____

Case III, v_{max} = _____

Case I: v_{max} = 3 volts

Case II: v_{max} = 6 volts

Case III: v_{max} = 12 volts

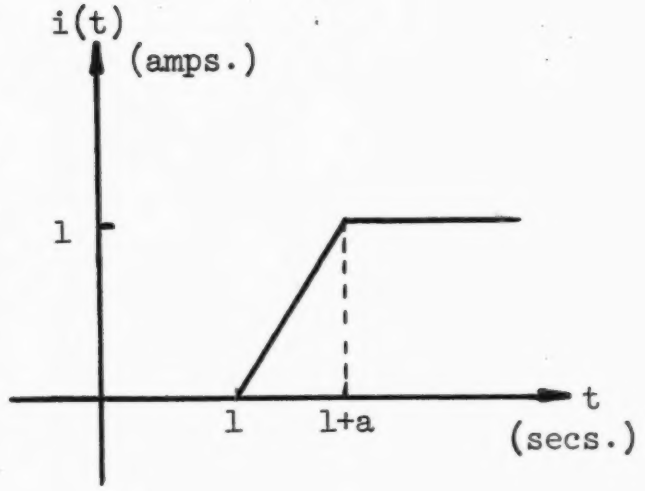


Fig. 12

An instantaneous current change is equivalent to a value of a = 0, in Fig. 12. It should be clear from the previous discussion that an instantaneous change in inductor current (a discontinuity in the current-time graph) is necessarily accompanied by an infinite value of inductor voltage.

Assuming thatyou could achieve an instantaneous current change, could an actual network elements (an inductor that could actually be built in a laboratory) withstand the effects of infinite voltage? (Write out your answer below.)

No actual element can withstand infinite voltage. One reason is that perfect insulation would be required.

This is equivalent to saying that an infinite pulse of voltage is physically unachievable.

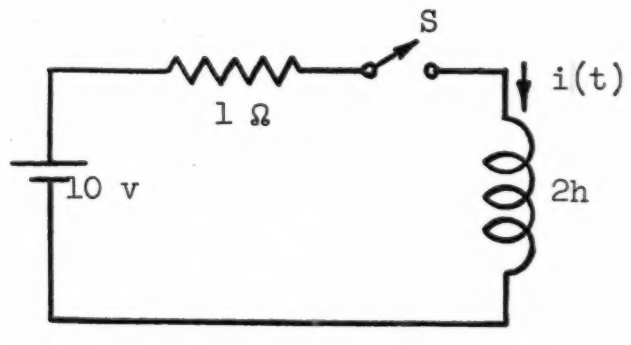


Fig. 13

Since the voltage across real circuit devices must be finite, ideal elements, used to represent the behavior of actual elements, should also be constrained to finite voltages. This leads us to the following restriction on the current in an inductor:

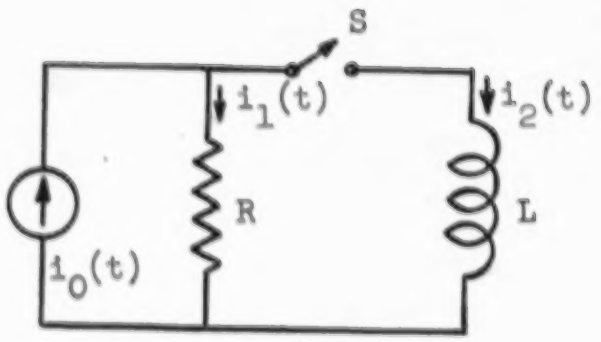
The current in an inductor must be a continuous function of time.

Consider the circuit of Fig. 13, in which the switch S closes at t = 0. What is the value of i(t) just before t = 0? (This time is often represented by t = (0-ϵ) or simply t = (0-). The current at this time is i(0-).)

Similarly, what is the value of i(0+), the current just after t = 0?

$i(0+) = 0$

This is necessary because any other value for $i(0+)$ would result in a discontinuity in the current and a consequent infinite voltage across the inductor.



$i_0(t) = 8(1+t)$

Fig. 14

As another illustration, consider the network of Fig. 14 in which the switch S is closed at t = 0.

Calculate $i_1(t)$ and $i_2(t)$ just after the switch is closed (i.e., determine $i_1(0+)$ and $i_2(0+)$).

$i_1(0+) = 8$ amperes

$i_2(0+) = 0$

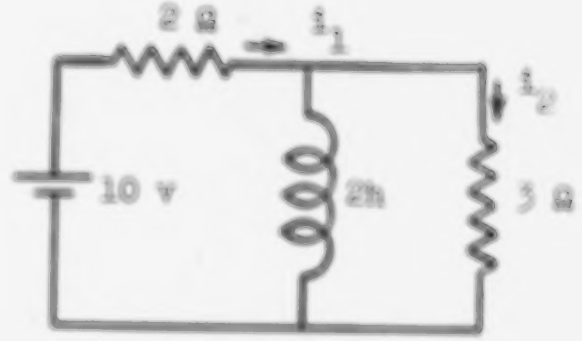


Fig. 15

There is a special and simple case that arises frequently when studying inductors. Suppose that the inductor current is constant, i.e., $di/dt = 0$. Then the inductor voltage is zero and the device is equivalent to a short circuit.

In a network containing inductors, resistors, and constant sources, the currents and voltage settle down after awhile to constant values. The network is then said to be in the "steady state".

Calculate the steady state values of i_1 and i_2 for the network of Fig. 15.

$i_1 = 5$ amperes

$i_2 = 0$

Remember that the steady state here means that currents and voltages are constants. Hence the inductor may be replaced by a short circuit.

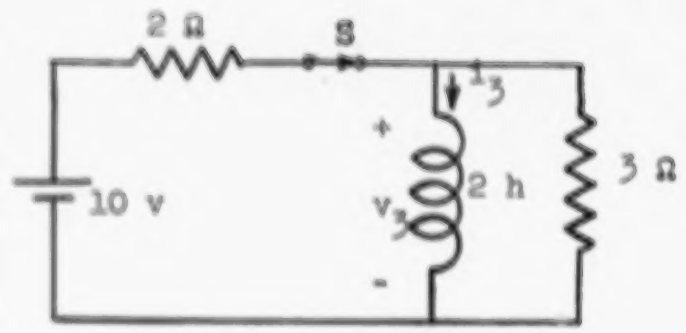


Fig. 16

The network of Fig. 16 is the same as that of Fig. 15, except for the switch S thathas been added.

Suppose the network is in the steady state before t = 0 and that S is opened at t = 0.

Determine the numerical value of:

a) $i_3(0+)$

b) $v_3(0+)$

c) di_3/dt at $t = 0+$

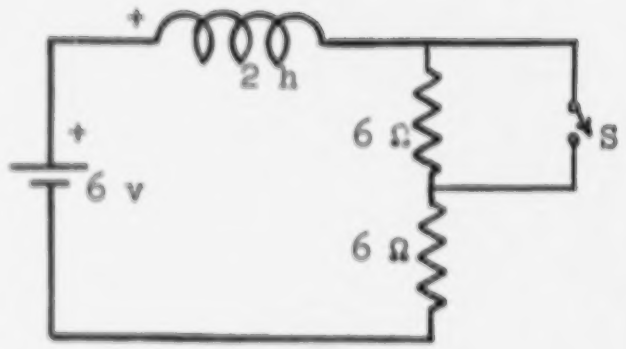
a) $i_3(0+) = 5$ amperes

b) $v_3(0+) = -5(3) = -15$ volts

c) $\frac{di_3}{dt} = \frac{v_3}{L} = \frac{-15}{2}$

$= -7.5$ amperes/sec.

(If you did not get these answers, consider the current in the 3-ohm resistance at $t = 0+$.)



Summary

Complete the following statements:

1) To insure finite voltage across an inductor, the inductor _____ must be a _____ function of time.
2) When switching takes place in a network containing inductors, the inductor currents just after the switching must _____.
3) An inductor having a constant current is equivalent to a _____ because its _____ is zero.
4) In a network of resistors, inductors, and constant sources, the steady state refers to the case in which all voltages and currents are _____.
5) In the figure on page 36, the network is in steady state when the switch S closes at t = 0. Determine the voltage across the inductor at t = (0-) and t = (0+).

Summary

1) ..... current must be a continuous function .....

2) ..... must equal, respectively, the inductor currents before switching.

3) ..... is equivalent to a short circuit because its voltage is zero.

4) ..... and currents are constants.

5) $v_L(0-) = 0$, $v_L(0+) = 3$ volts

Part III

Changing current and voltage in inductors

Since the voltage is related to the current in an inductor by differentiation, it is to be expected that integration of the voltage will yield the current.

For an inductor,

$$v = L\frac{di}{dt} \qquad (3)$$

Over a time interval beginning at t_1 and ending at t, each function in Eq. (3) must enclose the same area.

Express this fact mathematically.

$$\int_{t_1}^{t} v \, dt = L \int_{t_1}^{t} \frac{di}{dt} \, dt \qquad (4)$$

The integral on the right side of Eq. 4 can be reexpressed by changing the variable of integration from $\underline{t}$ to $\underline{i}$. This is accomplished by noting that

$$di = \frac{di}{dt}\, dt \text{ , and}$$

by making the required change in the integration limits.

Make these alterations in the integral on the right side of Eq. 4.

$$\int_{t_1}^{t} v\,dt = L \int_{i(t_1)}^{i(t)} di \qquad (5)$$

(If you are rusty on making variable changes in definite integrals, consult any introductory calculus book, a competent friend, or your instructor

$$\int_{t_1}^{t} v\,dt = L \int_{i(t_1)}^{i(t)} di$$

$$= L \left\langle i(t) - i(t_1) \right\rangle \qquad (6)$$

It is now a simple matter to evaluate the integral on the right side of Eq. 5. The result is Eq. 6.

Eq. 6 indicates that the difference between the initial and final current in an inductor is proportional to the integral of v(t) over that time interval or

$$i(t) - i(t_1) = \frac{1}{L} \int_{t_1}^{t} v \, dt \qquad (7)$$

Thus Eq. 7 is the inverse of the relationship, v = L di/dt. You should be able to duplicate this derivation at any time.

Calculate $i(t_2)$ for the case that L = 2 henries, $i(t_1) = 1$ ampere, $v(t) = e^{-t}$ $t_1 = 0$, and $t_2 = 0.10$ seconds.

$$i(t_2) - i(t_1) = \frac{1}{L}\int_{t_1}^{t_2} v(t)\,dt$$

$$i(0.10) - 1 = \frac{1}{2}\int_{0}^{0.10} e^{-t}\,dt$$

$$i(0.10) = \frac{1}{2}\,(3 - e^{-1/10}) \text{ amperes}$$

$$i(t) - i(t_1) = \frac{1}{L}\int_{t_1}^{t} v\,dt \qquad (7)$$

Consider a time interval that begins at t_1 and ends at t_2.

How is the area under the voltage curver over this interval related to the net change in the current during the same interval?

The area under the voltage curve is proportional to the net change in the current over the interval of time involved.

The proportionality factor is determined by the inductance.

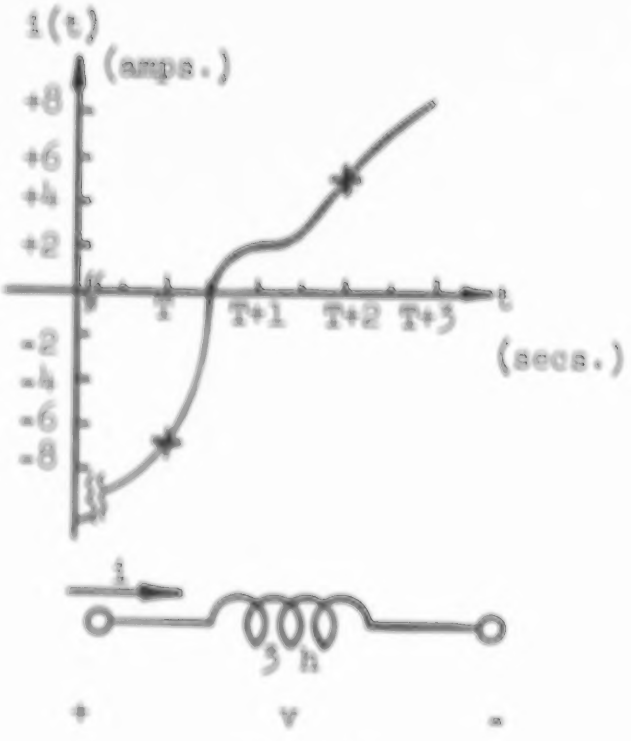


Fig. 17

In Fig. 17, the current in a 3-henry inductor is depicted. The current changes from -7 amperes to +5 amperes in a certain 2-second interval (beginning at t = T in Fig. 17).

Calculate the area enclosed by the inductor voltage curve in the same interval of time.

$$\int_{t_1}^{t_2} v\,dt = L\left\langle i(t_2) - i(t_1)\right\rangle$$

$$= 3\left\langle 5 - (-7)\right\rangle$$

$$= 36 \text{ volt seconds}$$

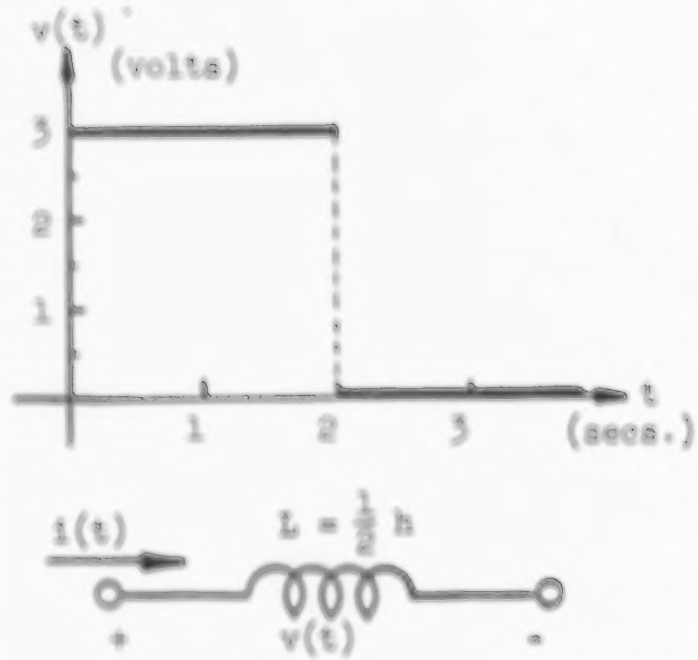


Fig. 18

Consider the function v(t) in Fig. 18. Suppose this is the voltage across a $\frac{1}{2}$ henry inductor whose current is 3 amperes at t = 0.

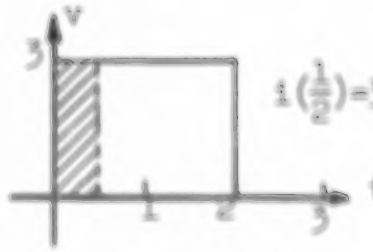
Remembering the expression for the change in current and interpreting the integral as an area, calculate:

a) the inductor current at t = $\frac{1}{2}$ sec.

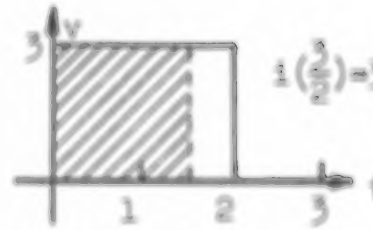
b) the inductor current at t = $\frac{3}{2}$ sec.

c) the inductor current at t = 3 sec.

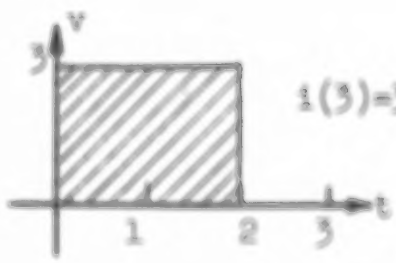
50
a)



$i(\frac{1}{2})=3 + 2(\frac{3}{2})=6a$



$i(\frac{3}{2})=3 + 2(\frac{9}{2})=12a$



$i(3)=3 + 2(6)=15a$

Fig. 19

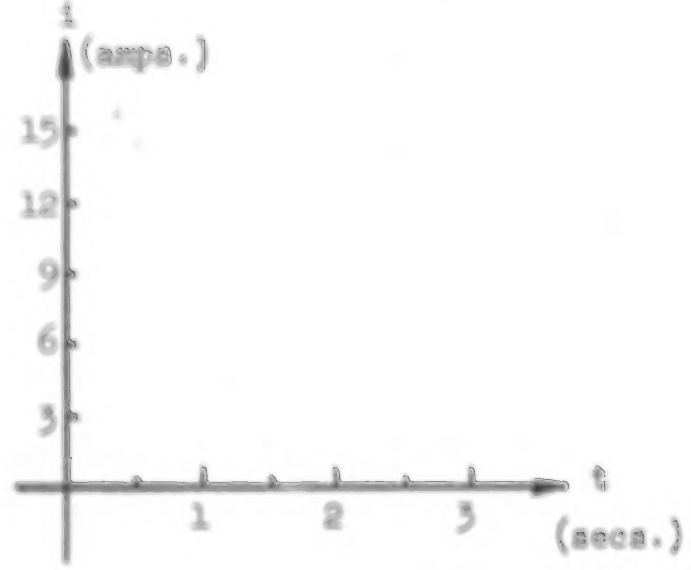
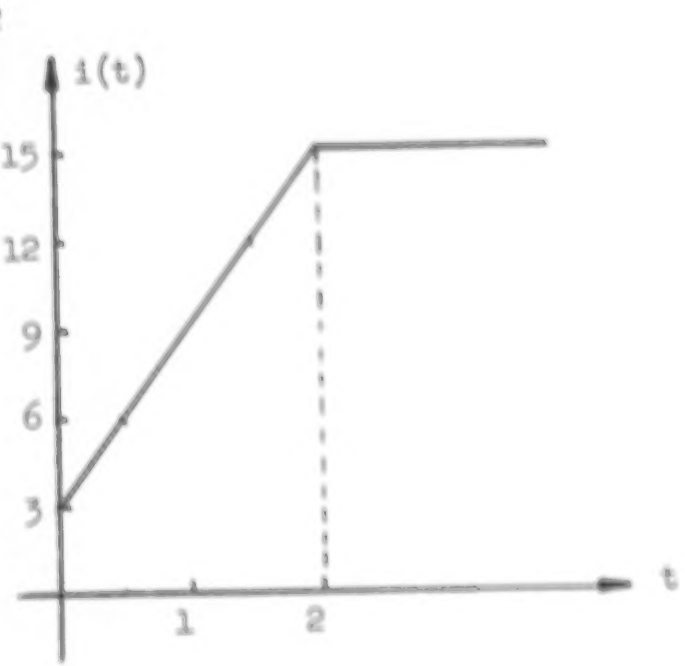


Fig. 20

In Fig. 19, the shaded sections indicate the accumulation of area as the integration proceeds. Corresponding to this increasing area, there is a build-up of current. After the time instant, t = 2, there is no further increase in area under the voltage plot and the current remains constant.

On Fig. 20, draw the current-time curve to show, graphically, the variation of current with time for the problem shown in Figs. 18 and 19.



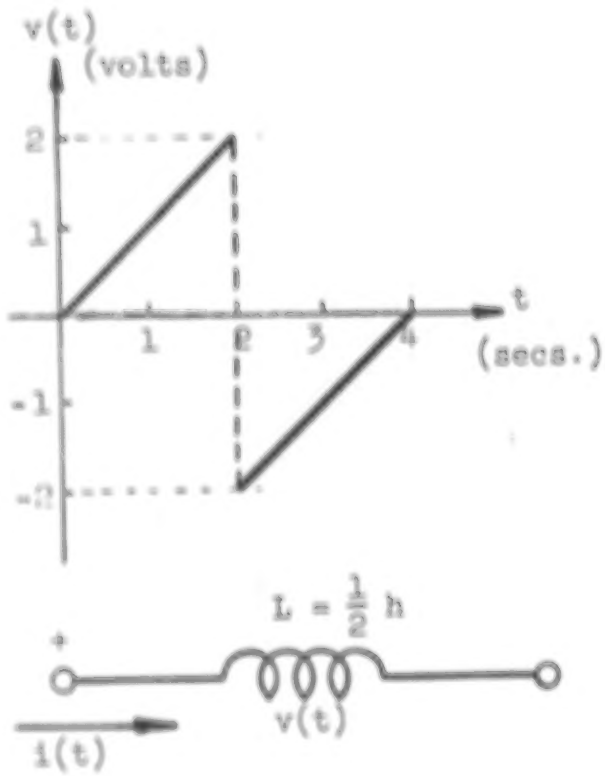


Fig. 21

Now, let's apply the same principles to a little different problem.

Let the voltage in Fig. 21 be across a $\frac{1}{2}$ henry inductor whose current is zero at t = 0. By estimating the area under the curve, determine the inductor current at the instants:

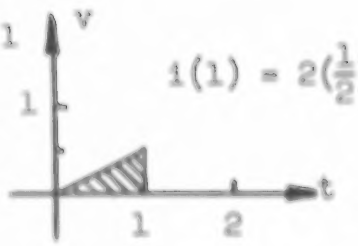
t = 1 sec.

t = 3 sec.

t = 4 sec.

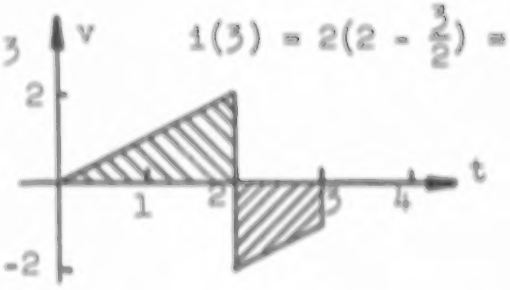
54

t = 1



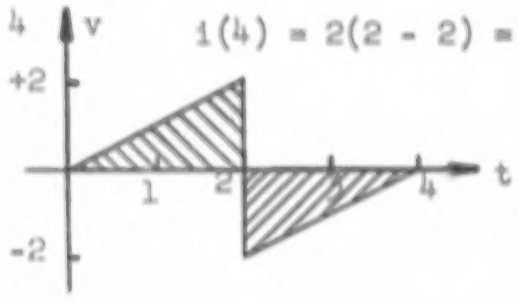
$i(1) = 2(\frac{1}{2}) = 1$ amp.

t = 3

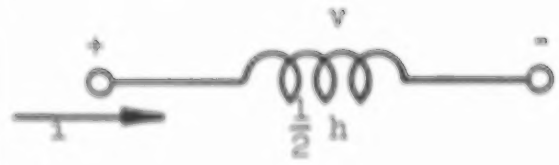


$i(3) = 2(2 - \frac{3}{2}) = 1$ amp.

t = 4



$i(4) = 2(2 - 2) = 0$



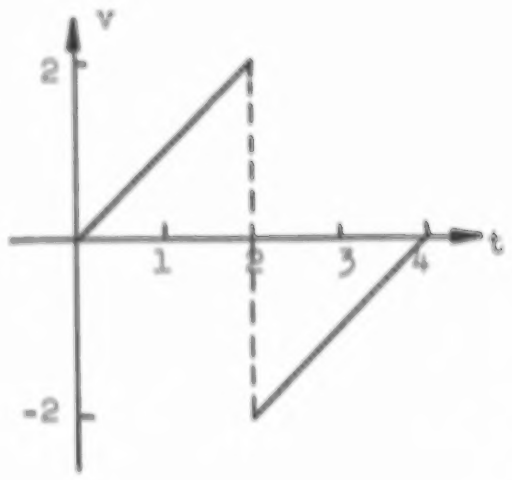


Fig. 21

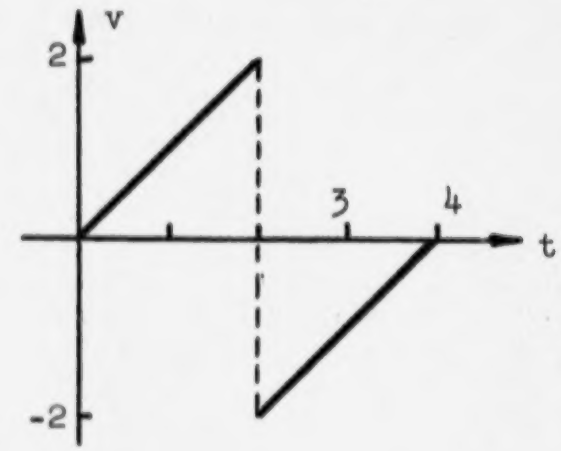
We continue to discuss the voltage v(t), Fig. 21. The inductor current, as a function of time during the interval from t = 0 until t = 2, can be determined by application of the inductor i-v relationship. To do this, the voltage during the interval must be expressed as a function of time.

Write the equation for v(t) during the interval from t = 0 until t = 2.

$v(t) = t \qquad 0 \leq t < 2$

<u>A Reminder</u>

$$i(t) - i(t_1) = \frac{1}{L} \int_{t_1}^{t} v(t)\, dt$$



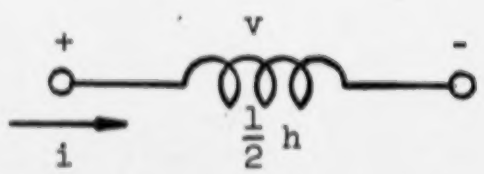


Fig. 21 (repeated)

On the next few pages you will calculate and plot the current i(t) first for one of these time intervals and then the other. (Note that each interval has a different voltage equation.)

Determine the equation for i(t) for $0 \leq t \leq 2$.

Sketch this function.

$$i(t) - i(0) = 2\int_0^t t\,dt$$

but $i(0) = 0$,

$$i(t) = t^2 \qquad 0 \le t \le 2 \qquad (8)$$

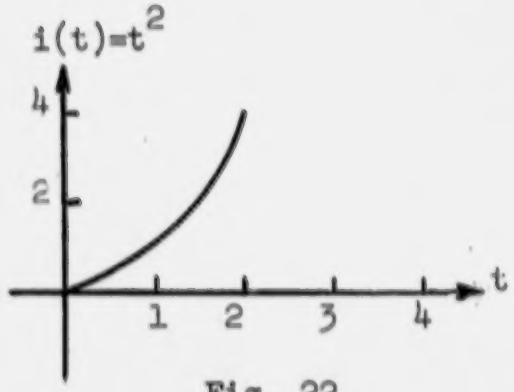
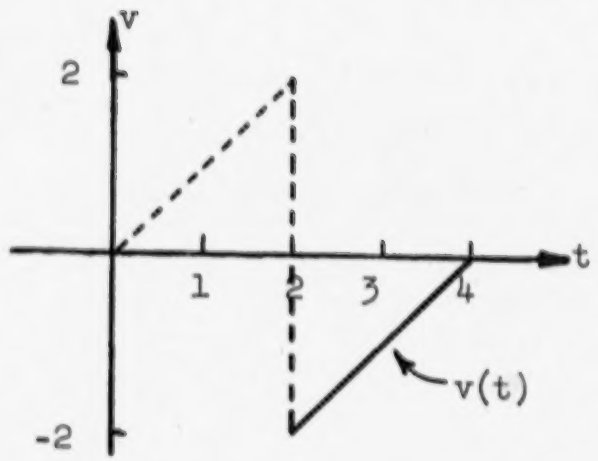


Fig. 22

The current at t = 2 is i(2) = 4 amperes, as indicated by Eq. 8.

The same procedure can be applied again to calculate i(t) for the interval from t = 2 until t = 4. The lower limit is now chosen at $t_1 = 2$. The equation for v(t), Fig. 21, for $2 < t < 4$ is required.

Determine the equation for v(t) for $2 < t < 4$.



$$v(t) = t - 4 \qquad 2 \leq t \leq 4 \qquad (9)$$

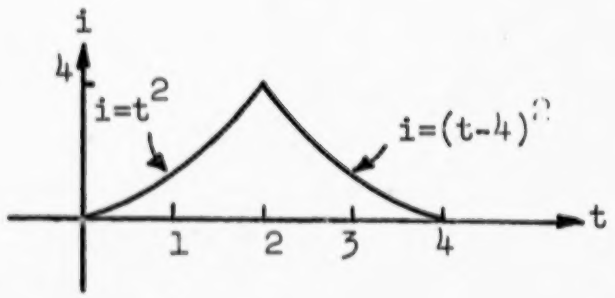
Using this equation for v(t) for the interval $2 \leq t \leq 4$, calculate an expression for i(t) in this same interval.

Complete the sketch for i(t) from $0 \leq t \leq 4$ that you began on page 57 (or Fig. 22).

$$i(t) - i(2) = 2\int_2^t (t-4)dt$$

$$i(t) = 4 + 2(\tfrac{1}{2}t^2 - 4t)\Big|_2^t$$

$$i(t) = (t-4)^2 \text{ for } 2 \leq t \leq 4$$



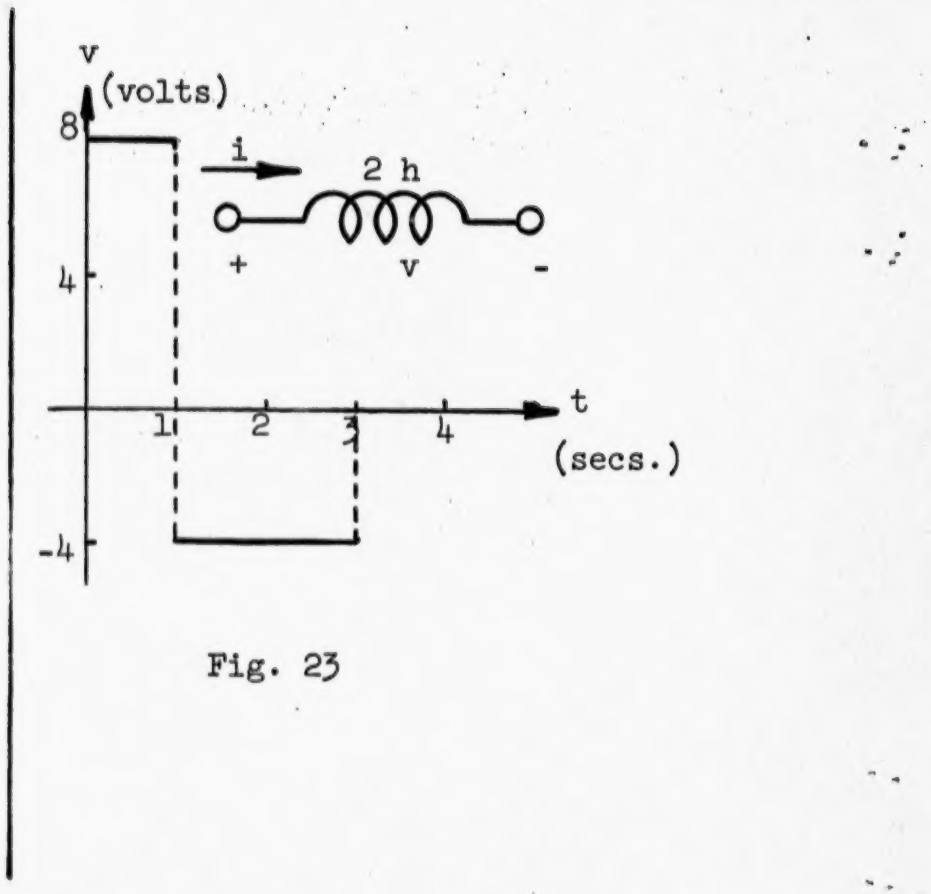


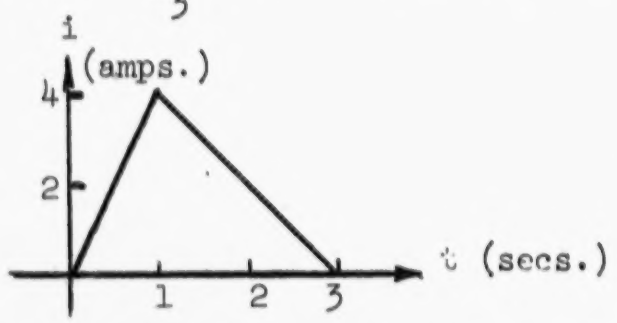
Fig. 23

Suppose the voltage across a 2-henry inductor varies according to Fig. 8. The current is zero at $t = 0$. There are three time intervals of interest, $0 \leq t \leq 1$, $1 \leq t \leq 3$, $t \geq 3$. Calculate $i(t)$ for each interval, and sketch $i(t)$ for $t \geq 0$. (Hint: Apply your knowledge of the basic i-v relationships in an inductor to each of the three time intervals, successively, and then combine the results.)

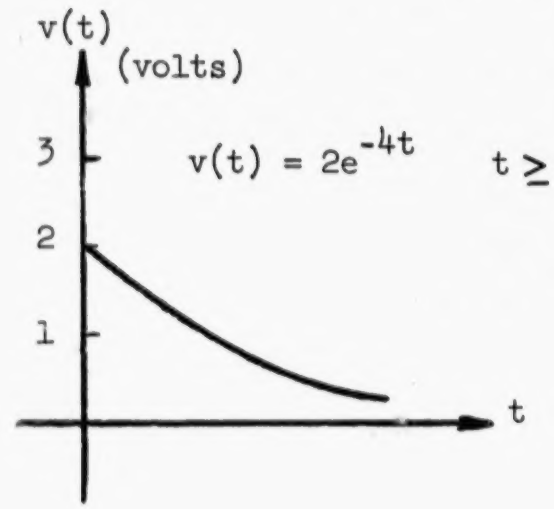
$$i(t) = \frac{1}{2}\int_0^t 8\,dt = 4t \quad \text{for } 0 \le t \le 1$$

$$i(t) = 4 + \frac{1}{2}\int_1^t -4\,dt = 6 - 2t \text{ for } 1 \le t \le 3$$

$$i(t) = 0 + \int_3^t 0\,dt = 0 \quad \text{for } t \ge 3$$



Note that this is the reverse of the problem given on page 7.



$v(t) = 2e^{-4t} \qquad t \ge 0$

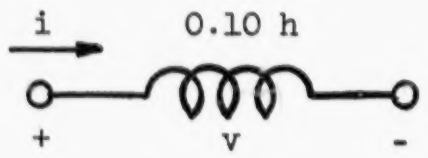


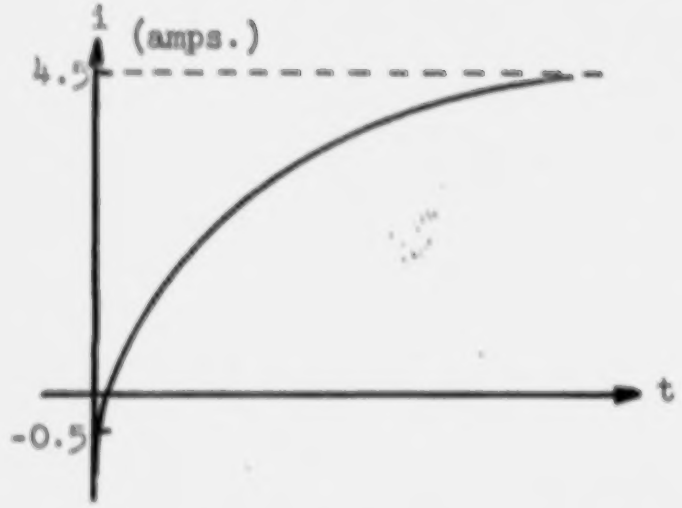
Fig. 24

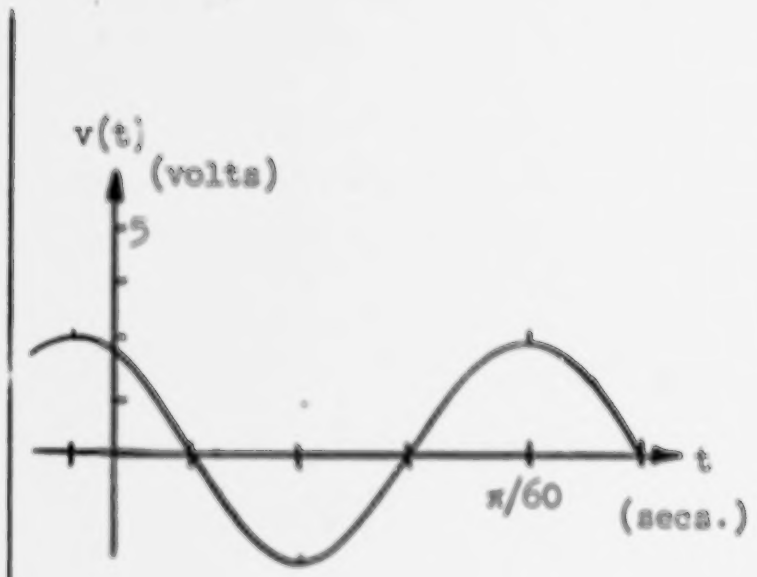
In a certain 0.10 h inductor the current is -0.30 ampere at t = 0. The variation of the voltage is indicated in Fig. 24.

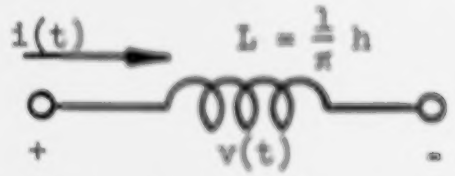
Determine the equation of the inductor current for $t \geq 0$, and sketch.

$$i(t) = -\frac{1}{2} + 10\int_0^t 2e^{-4t}\,dt$$

$$i(t) = 4.5 - 5e^{-4t} \text{ for } t \geq 0$$







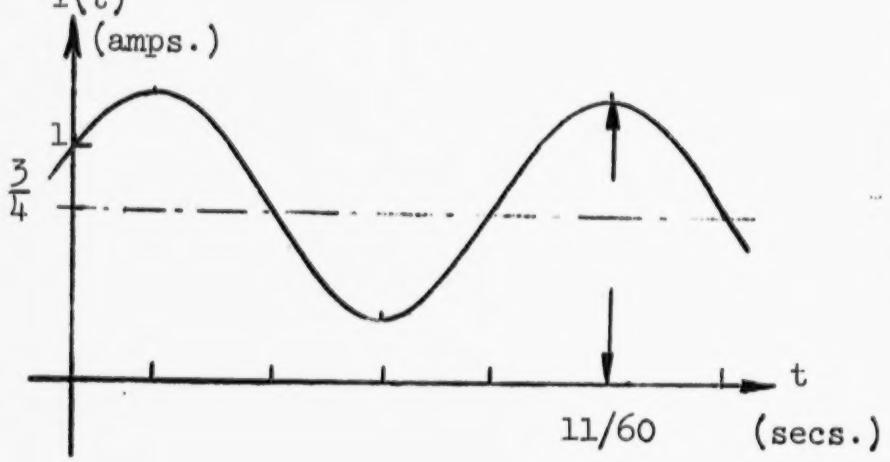
i(1) = 1 amp.

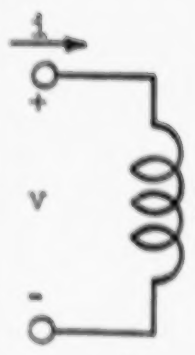
Fig. 25

As a final example, consider the inductor of Fig. 25. In this case, the current at t = 1 is 1 ampere, and the inductor voltage is $v(t) = 5 \cos \pi(10t+\frac{1}{6})$. Determine the equation for i(t) and sketch.

$$i(t) = 1 + \pi \int_1^t f \cos \pi(10t + \tfrac{1}{6})\, dt$$

$$i(t) = \tfrac{3}{4} + \tfrac{1}{2} \sin \pi(10t + \tfrac{1}{6})$$





$$i(t) - i(t_1) = \frac{1}{L} \int_{t_1}^{t} v\, dt$$

(a)

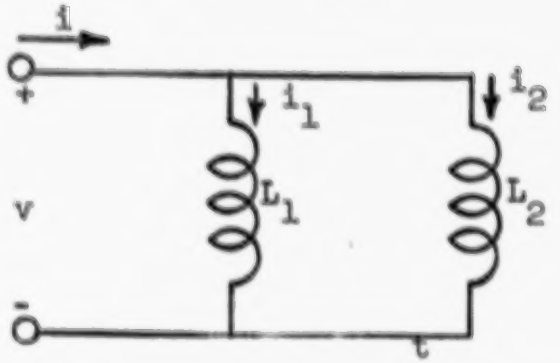


Fig. 26

$$i_1(t) - i_1(t_1) = \frac{1}{L_1} \int_{t_1}^{t} v_1\, dt$$

$$i_2(t) - i_2(t_1) = \frac{1}{L_2} \int_{t_1}^{t} v_2\, dt$$

(b)

In Fig. 26b, two inductors, L_1 and L_2, are connected in parallel. The voltage-current relation for each element in Fig. 26b is also indicated. The question of immediate interest is how to choose L in Fig. 26a so that it is equivalent to the parallel combination of L_1 and L_2 of Fig. 26b.

By combining the voltage-current relation for L_1 and L_2, noting the parallel connection means $v_1 \equiv v_2$, write the relation between $\underline{v}$ and $\underline{i}$ in Fig. 26b.

$$i = i_1 + i_2$$

$$v = v_1 = v_2$$

$$i_1(t) + i_2(t) - \left\{ i_1(t_1) + i_2(t_1) \right\} = \left(\frac{1}{L_1} + \frac{1}{L_2} \right) \int_{t_1}^{t} v \, dt$$

or

$$i(t) - i(t_1) = \left(\frac{1}{L_1} + \frac{1}{L_2} \right) \int_{t_1}^{t} v \, dt \qquad (10)$$

Compare the voltage-current relation for the single inductor of Fig. 26a. with the voltage-current relation for L_1 and L_2 in parallel as given by Eq. 10 (page 68).

If the single inductor L is equivalent to the parallel connection of L_1 and L_2, how are L, L_1, and L_2 related?

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}$$

or

$$L = \frac{L_1 L_2}{L_1 + L_2}$$

Note that the rule for combining inductors in parallel is the same as that for resistors in parallel.

Summary

Complete the following statements:

1) Change in inductor current is proportional to the change in _____ _____.

2) In a $\frac{1}{2}$ henry inductor the current changes from -1 to +2 amperes in a 2-second interval. The corresponding area enclosed by the voltage curve is _____ (units?).

3) In question 2) above the average inductor voltage during the 2-second interval is _____ volts.

4) In a 2-henry inductor, i(0) = 0. When will the current be 6 amperes if the voltage is held at a constant 1 volt?

5) The parallel connection of a 10-henry and an L-henry inductor is equivalent to a 6-henry inductor. L = _____ henry(s).

1) ..... to the change in the <u>area enclosed by</u> the inductor voltage.
2) 1 volt second
3) 1/2 volt
4) 12 seconds
5) 15 henries

Part IV

Energy in an Inductor

Consider the expression for the instantaneous power to an inductor.

$$p = vi = Li\frac{di}{dt} \tag{11}$$

The last term in Eq. 11 can be rewritten as the time derivative of a certain function.

Rewrite Eq. 11 in the following form: (fill in the bracket with the correct function).

$$p = \frac{d}{dt}\left\{\qquad\right\}$$

$$p = \frac{d}{dt} \left\langle \frac{1}{2} L\, i^2 \right\rangle \qquad (12)$$

We are interested in calculating the energy increment, Δw, that is received by an inductor over a time interval from $t = t_1$ until $t = t_2$. Since power is the rate of change of energy, integration of the power should yield Δw.

Express Δw in terms of an integral of the instantaneous power to an inductor.

$$\Delta w = \int_{t_1}^{t_2} p \, dt \qquad (13)$$

or

$$\Delta w = \int_{t_1}^{t_2} Li \frac{di}{dt} \, dt$$

Substitution of Eq. 12 into Eq. 13 yields

$$\Delta w = \int_{t_1}^{t_2} \frac{d}{dt} \left\{ \frac{1}{2} Li^2 \right\} dt \qquad (14)$$

It is now easy to evaluate the right side of Eq. 14.

Complete the development for Δw, by evaluating the right side of Eq. 14.

$$\Delta w = \int_a^b d \left\{ \frac{1}{2} L\, i^2 \right\} \quad \text{where} \quad \begin{cases} a = \frac{1}{2} L\, i^2(t_1) \\ b = \frac{1}{2} L\, i^2(t_2) \end{cases}$$

or

$$\Delta w = \frac{1}{2} L \left\{ i^2(t_2) - i^2(t_1) \right\} \qquad (15)$$

Note that the energy increment, Δw, depends on the square of the current and is thus independent of the current direction.

Furthermore, Δw may be positive, negative, or zero.

State the conditions under which Δw is positive.

The energy received by an inductor, Δw, over a time interval from $t = t_1$ until $t = t_2$ will be positive when $|i(t_2)|$ is greater than $|i(t_1)|$ (and vice versa).

$$\Delta w = \frac{1}{2} L \left\{ i^2(t_2) - i^2(t_1) \right\} \quad (15)$$

The expression for Δw indicates that an increase in the magnitude of the current is accompanied by a flow of energy into the inductor. Likewise, a decrease in current magnitude results in an actual outflow of energy from the inductor.

Thus the inductor acts as an energy reservoir, sometimes receiving and other times supplying energy. The energy of the inductor is associated with the magnetic field that is a result of the current.

Consequently, when the inductor current is zero, there is no magnetic field and the inductor energy is _____.

..... no magnetic field and the inductor energy is <u>zero</u>.

The total (or net) energy received by an inductor can be determined by calculating the energy increment, starting at a time instant when the current is zero.

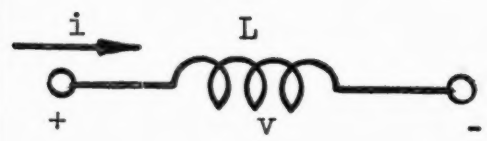
Thus if $i(t_1) = 0$, then $w = \Delta w$, where w is the total energy in the inductor. In this case, the total energy can be expressed as

$$w = \text{_____}. \tag{16}$$

$$w = \frac{1}{2} L\, i^2(t_2) \qquad (16)$$

The subscript 2 on $\underline{t}$ in Eq. 16 is unnecessary since t_2 can be any instant whatever. Thus the total stored energy in an inductor whose current is $\underline{i}$ amperes is

$$w = \frac{1}{2} L\, i^2 \qquad (17)$$



$$w = \frac{1}{2} Li^2$$

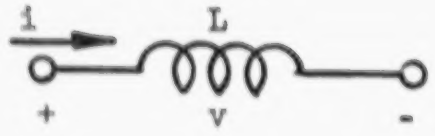
The energy in a certain inductor is 3 joules, when the current is 3 amperes.

What is the inductance?

L = _____ henry.

$3 = \frac{1}{2} L(3)^2$

or $L = \frac{2}{3}$ henry



$$w = \frac{1}{2} Li^2$$

A current change from -4 amperes to +6 amperes in a certain inductor requires an energy transfer of 1 joule.

Calculate the inductance. What is the direction of energy transfer?

$$\Delta w = \tfrac{1}{2} L \left\langle i^2(t_2) - i^2(t_1) \right\rangle$$

$$1 = \tfrac{1}{2} L \left\langle (6)^2 - (-4)^2 \right\rangle$$

$$L = \tfrac{1}{10} \text{ henry}$$

Energy is transferred into the inductor.

$$\Delta w = \tfrac{1}{2} L \left\langle i^2(t_2) - i^2(t_1) \right\rangle \qquad (15)$$

You have derived and used an expression for the relationship between the energy transferred in or out of an inductor and its current changes. From memory, write that expression.

Now, there is one additional feature about this relationship that deserves emphasis. Write your answer to the question,

"To what extent does Δw depend upon the detailed manner in which $i(t)$ varies from $i(t_1)$ to $i(t_2)$?"

The equation for Δw indicates that the energy increment depends only on the initial and final values of the current. It is not related to the intermediate values of current assumed by $i(t)$ as it varies from $i(t_1)$ to $i(t_2)$.

Summary

Complete the following statements:

1) Doubling the magnitude of the current in an inductor will _____ the energy stored in the inductor.

2) Increasing the current by 1 ampere in a 2-henry inductor causes the stored energy to decrease by 3 joules. The initial and final values of the current must have been: $i(t_1)$ = _____ amperes; $i(t_2)$ = _____ amperes.

3) The current at t = 0 in a 2-henry inductor is 2 amperes. It is also 2 amperes at t = 10 seconds. The total energy received by the inductor in this interval is _____ joules.

4) The energy stored in an inductor is recoverable; it is released whenever _____.

1) quadruple

2) $i(t_1) = -2$ amperes

   $i(t_2) = -1$ ampere

3) zero

4) the current magnitude decreases.

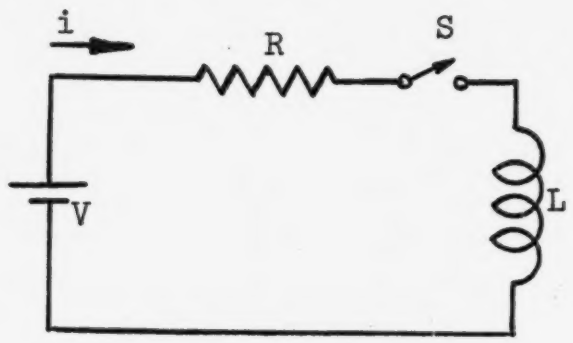


Fig. 27

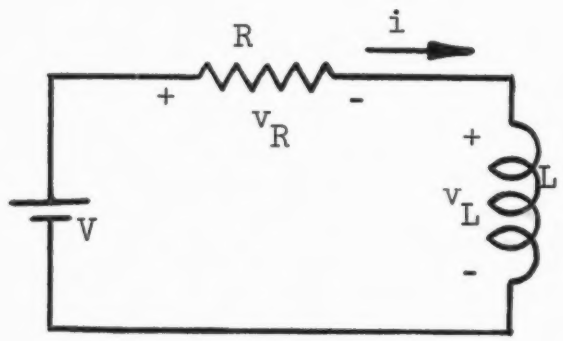
Part V

Solving for the current

Consider the network of Fig. 27. The switch, S, closes at t = 0. Therefore, since the inductor current cannot change abruptly, the current at t = 0+ is zero.

For $t \geq 0$, you use the same approach that you used in solving resistive networks; apply Kvl and the i-v relationship for each component. The loop equation will then involve a derivative -- it will be a differential equation.

Write the differential equation for the network.



$$v_L + v_R = V$$

or

$$L \frac{di}{dt} + Ri = V \qquad (18)$$

Earlier in this study (see page 33) a type of steady state for inductor networks was considered. That steady state was characterized by constant inductor current and zero inductor voltage.

Consider the differential equation you just wrote. Is there any constant value for i that will satisfy this equation? If there is, determine its value in terms of V, L, R.

$$i = \frac{V}{R} \qquad (19)$$

This value of the current will satisfy the differential equation, Eq. 18.

(The above answer is easily determined by noting that if i is constant, $\frac{di}{dt}$ is zero. This reduces Eq. 18 to:

$Ri = V$.)

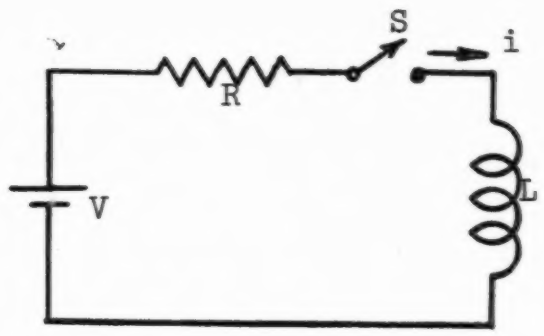


Fig. 27

$$L \frac{di}{dt} + Ri = V \qquad (18)$$

$i(0+) = 0$

Equation 19 gives the value of a steady (or constant) current that can exist in the network of Fig. 27 for $t > 0$. To get to this value, the current must undergo a transition from $i = 0$ when S is closed to $i = V/R$ at some later time. Since the current cannot change abruptly, some time interval is required for the transition to take place.

Let $i(t)$ for $t \geq 0$ be the function that describes the current in the circuit during the time of transition. What <u>two</u> characteristics of $i(t)$ do you now know (two conditions that must be satisfied by the expression for current when $t \geq 0$)?

a) $i(0+) = 0$

b) $i(t)$ for $t \geq 0$ must satisfy Eq. 18.

$$L \frac{di}{dt} + Ri = V \qquad (18)$$

$i(0+) = 0$

To determine i(t) for t ≥ , the differential equation, Eq. 18, must be solved. In particular, the so tion must satisfy i(0+) = 0. It remains to be seen whether the steady stat i.e., i = V/R, is actually achieved. (Note that i = V/R has only bee shown to be a possible solution.)

The solution of Eq. 18 fo i(t) can be started through a multiplication by dt and the separation of i di terms from t and dt terms.

Carry out this separation, riting all i and di terms on the left side and t and dt terms on the right side.

$$L\,di + Ri\,dt = V\,dt$$

$$L\,di = (V-Ri)\,dt$$

$$\frac{di}{i - \frac{V}{R}} = -\frac{R}{L}\,dt \qquad (19)$$

$$\therefore i(0+) = 0$$

oth sides of Eq. 19 may now be formed and the
(t).
of Eq. 19 and solve for i(t).

The indefinite integral
resulting expression solved f
Form the indefinite inte

$$\ln\left(i - \frac{V}{R}\right) = -\frac{R}{L}t + K_1$$

(K_1 is the constant of integration)

$$i - \frac{V}{R} = e^{K_1} e^{-(R/L)t}$$

Let $K = e^{K_1}$

$$i(t) = \frac{V}{R} + K e^{-(R/L)t} \qquad (20)$$

Equation 20 for i(t) contains an undetermined constant, K. However, the condition i(0+) = 0 has not yet been imposed.

Determine the value of K in Eq. 20 so that i(0+) = 0.

$i(0+) = 0 = \frac{V}{R} + K$

so $K = -\frac{V}{R}$ and

$$i(t) = \frac{V}{R}\left(1 - e^{-(R/L)t}\right) \qquad (21)$$

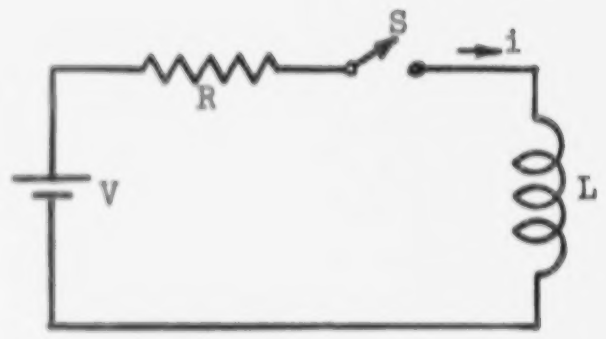


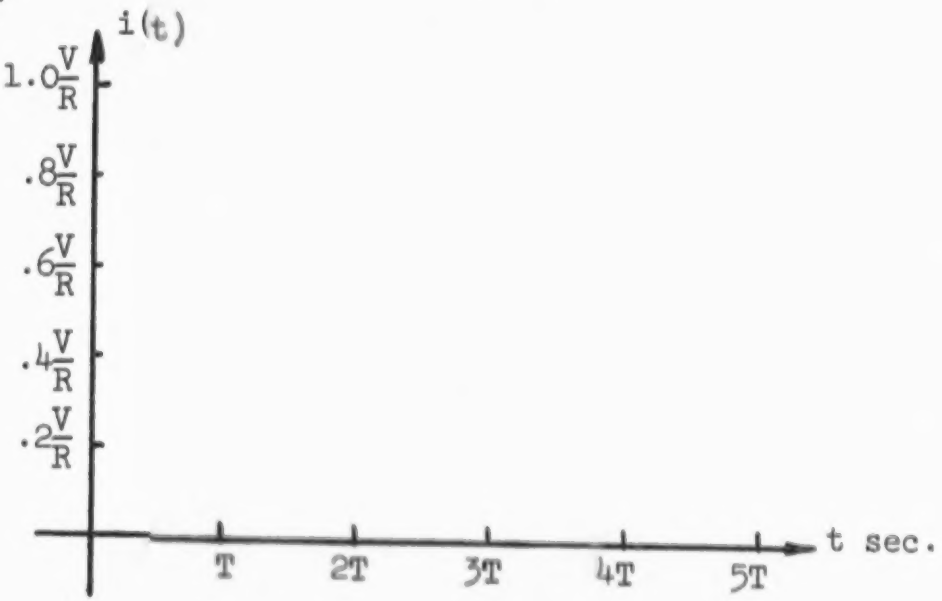
Fig. 27

The variation of the current for $t \geq 0$ is given in Eq. 21. It is instructive to plot this function for the values t = T, 2T, 3T, etc., where $T = \frac{L}{R}$. T is called the time constant of the network.

Using tabulated exponential values, construct an accurate sketch of i(t) for t = T, 2T, 3T, etc.

Table of Values

x	e^{-x}
1	.368
2	.135
3	.050
4	.018
5	.007



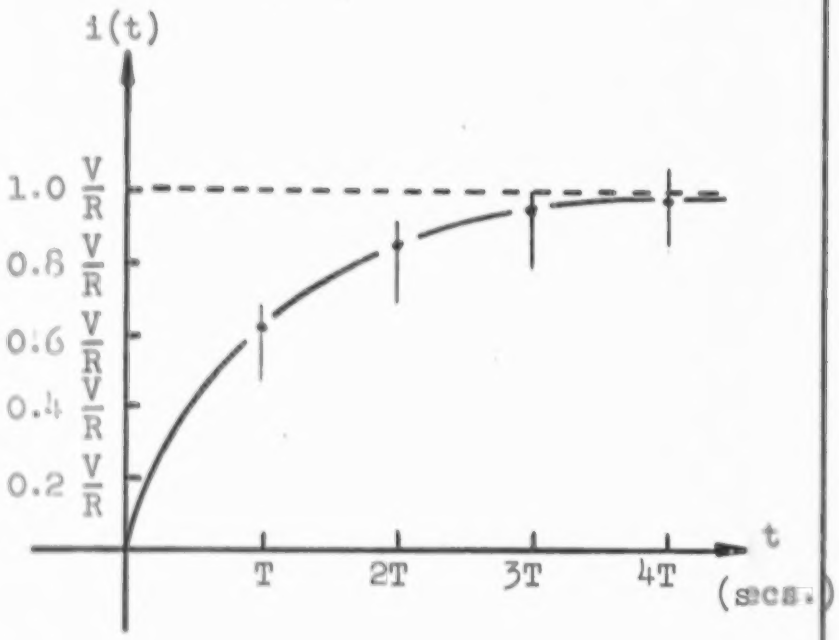


Fig. 28

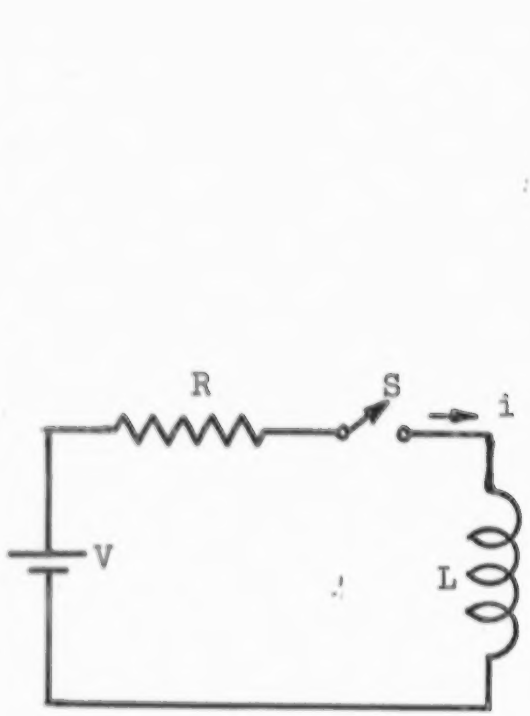


Fig. 27 (repeated)

Fig. 28 depicts i(t) for $t \geq 0$. It is clear that the steady state is approached, i.e., after awhile i(t) becomes essentially constant, equal to V/R amperes.

Theoretically, it takes an infinite time for i(t) to reach V/R. As a practical matter however, $e^{-(Rt/L)}$ is essentially zero for $t \geq 5T$. Thus the transition period lasts about 5 time constants.

Suppose in Fig. 27: V = 10 volts, R = 2 ohms, L = 4 henries. Determine:

i) the steady state current

ii) the time constant

iii) duration of transition period

i) $\frac{V}{R}$ = 5 amperes

ii) $T = \frac{L}{R}$ = 2 seconds

iii) 5T = 10 seconds

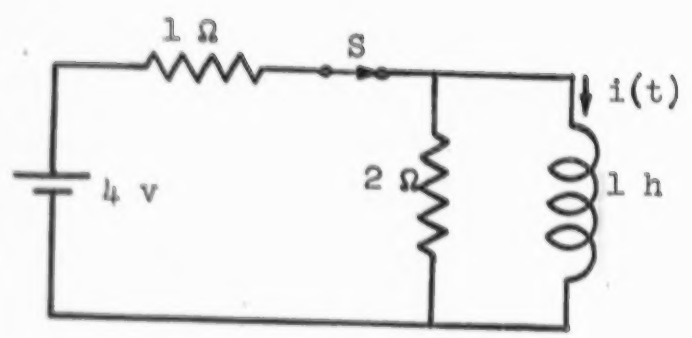
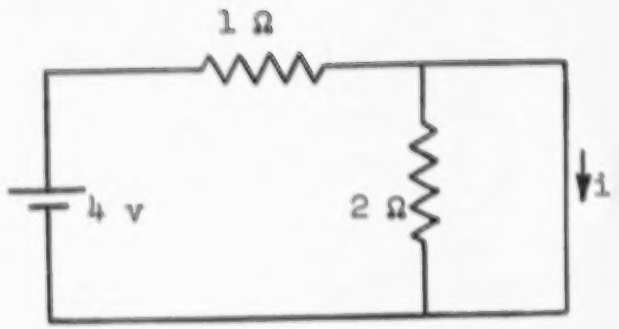


Fig. 29

As a second example of an R-L network problem, consider Fig. 29. The switch S has been closed for a long time so that at t = 0-, the network has achieved steady state. At t = 0, S is opened. The problem is to determine i(t) for $t \geq 0$.

As before, the problem requires solving a differential equation, as well as satisfying some initial condition.

Calculate i(0-), the steady state inductor current just before t = 0.



$i(0-) = \frac{4}{1} = 4$ amperes

(Remember that L is a short circuit for steady currents.)

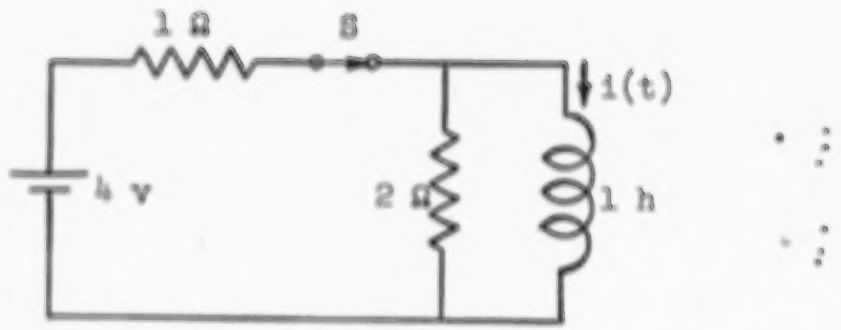


Fig. 29

After t = 0, the network of Fig. 29 simplifies as a result of S opening.

Draw the simplified network that determines i(t) for $t \geq 0$. What is the initial value of i(t), (i.e., i(0+)?

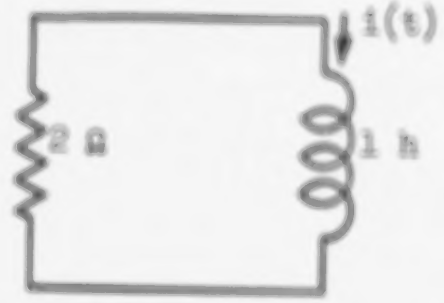


Fig. 30

$i(0+) = i(0-) = 4$ amperes

(Remember that the current in an inductor cannot change abruptly.)

Write the equation that i(t) in Fig. 30 must satisfy. (Hint: KVL).

$$\frac{di}{dt} + 2i = 0 \qquad (22)$$

$i(0+) = 4$ amperes.

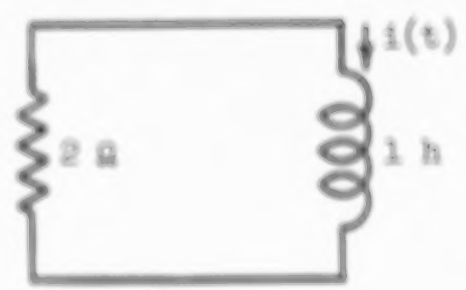


Fig. 30

The network of Fig. 29 was in the steady state when S was opened. The new network, Fig. 30, will seek a new steady state.

Using Eq. 22, determine the steady state current for the network of Fig. 30.

$$\frac{di}{dt} + 2i = 0 \qquad (22)$$

In steady state $\frac{di}{dt} = 0$

Thus i = 0 in steady state.

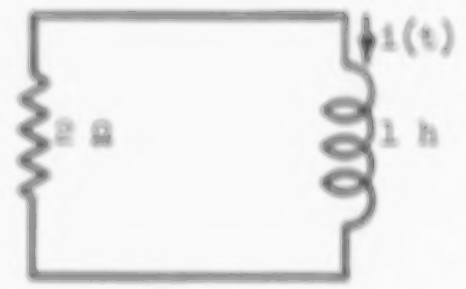


Fig. 30

The function i(t), Fig. 30, has an initial value of 4 amperes, and a final value of zero amperes. (We often refer to the steady state current as the final value.)

To determine i(t) for $t \geq 0$, proceed as before:

i) Separate Eq. 22

ii) Integrate

iii) Select integration constant so that i(0+) = 4.

Do these three things now.

i) $\frac{di}{i} = -2dt$

ii) $\ln i = -2t + K_1$

$$i = e^{-2t+K_1} = K e^{-2t}$$

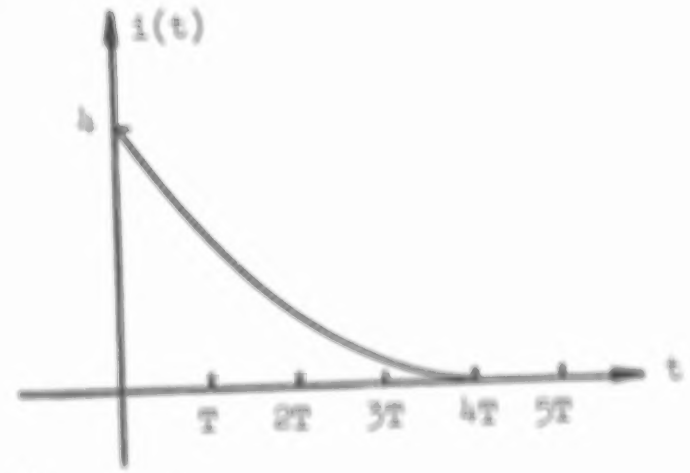
where $K = e^{K_1}$

iii) $i = K e^{-2t}$

$i(0+) = 4 = K$

$i = 4 e^{-2t}$ (23)

To complete this problem, sketch i(t), Eq. 23, and state the numerical value of the time constant, T.



$T = \frac{1}{2}$ second.

Summary

Complete the following statements:

1) The equilibrium equation governing the current in an R-L network is a _____ equation.
2) The solution of the equilibrium equation is completely determined by three numbers: i) the _____ value of the current; ii) the _____ value of the current; and iii) the _____ constant.
3) The steady state current in an R-L network is determined by solving for the current with the inductor replaced by a _____ _____.
4) In a series connection of a resistor and an inductor, the time constant equals _____.

1) ..... is a <u>differential</u> equation .....

2) i) ..... <u>initial</u> value of the current.

ii) ..... final (or steady state) value of the current.

iii) ..... <u>time</u> constant

3) ..... by a <u>short circuit</u>.

4) $T = \frac{L}{R}$

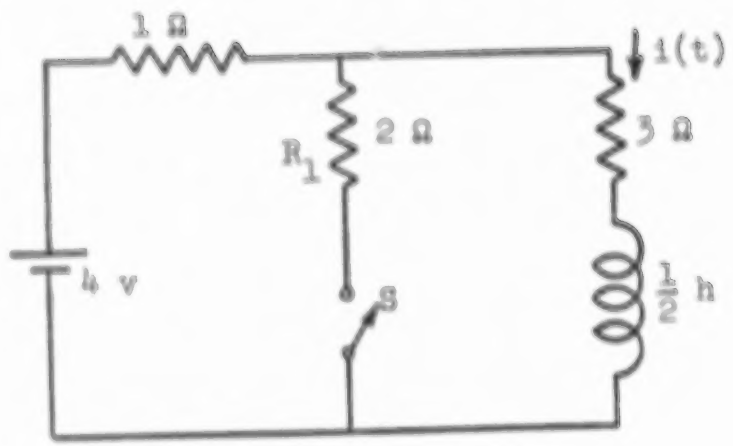


Fig. 31

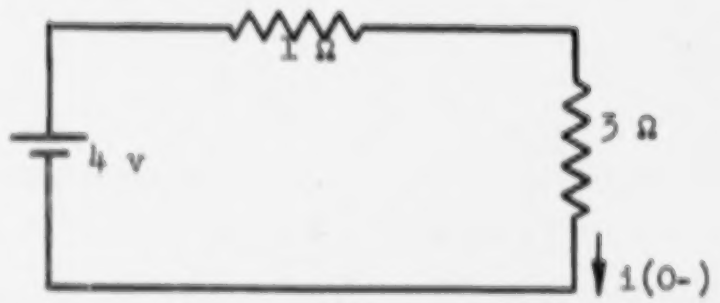
Part VI

Review

For a final example, consider the network of Fig. 31. This network is in the steady state before t = 0. At t = 0, S is closed, causing the inductor current to change to a new steady state value. The problem is to determine i(t) for $t \geq 0$.

Begin by calculating i(0-), the steady state current before the S is closed.

126



Note:

1) Before t = 0 R_1 is not in the network and thus is ignored.
2) In the steady state, L is a short circuit.

$$i(0-) = \frac{4}{4} = 1 \text{ ampere.}$$

What is the initial value of i(t) after S is closed (i.e., i(0+)?

Why?

i(0+) = 1 ampere

i(0+) = i(0-) because the current in an inductor cannot change abruptly.

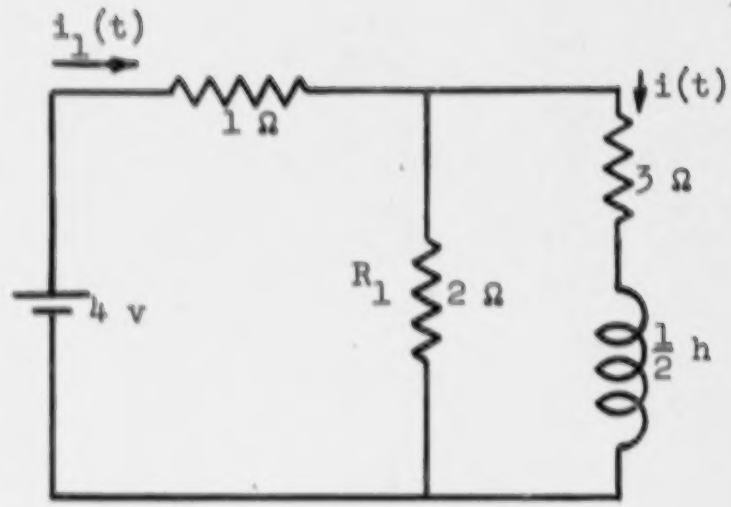


Fig. 32

For $t \geq 0$, S is closed and the network becomes that of Fig. 32. Let us construct the equilibrium equation for $i(t)$, $t \geq 0$.

Observe that the voltage across R_1 can be written in terms of i and $\frac{di}{dt}$ as:

$$v_{R_1} = 3i + \frac{1}{2}\frac{di}{dt} \quad . \tag{24}$$

Express the current, $i_1(t)$, in the 1-ohm resistor in terms of i and $\frac{di}{dt}$.

$$i_1 = \frac{5}{2}\, i(t) + \frac{1}{4}\frac{di}{dt} \qquad (25)$$

Hint: $i_1(t)$ equals the sum of i(t) and the current in R_1.

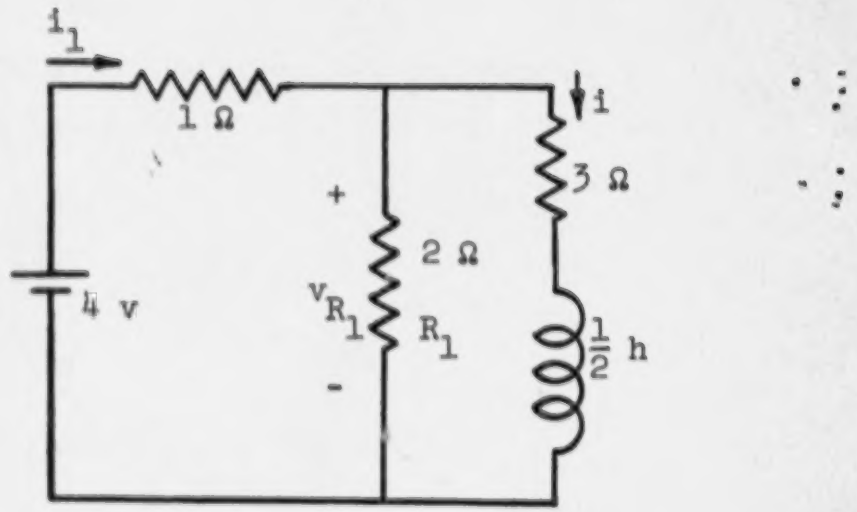


Fig. 32

The voltage across the 1-ohm resistor equals $4 - v_{R_1}$. Also, since the resistance equals 1 ohm, we have:

$$4 - v_{R_1} = 1\, i_1 \,. \tag{26}$$

Substituting Eqs. 24 and 25 into Eq. 26 yields the equilibrium equation for i(t).

Carry out this substitution and simplify.

$$4 - \left\langle 3i + \frac{1}{2}\frac{di}{dt} \right\rangle = \frac{5}{2} i + \frac{1}{4}\frac{di}{dt}$$

or

$$\frac{di}{dt} + \frac{22}{3} i = \frac{16}{3} \qquad (27)$$

$i(0+) = 1$

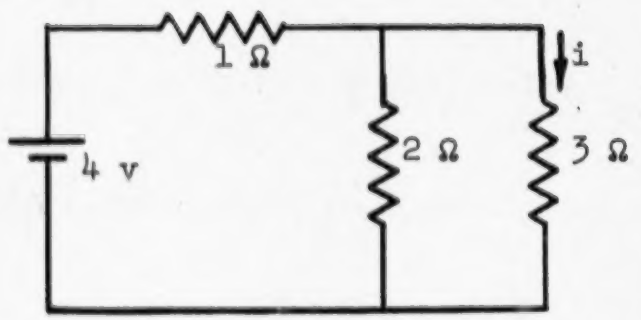


Fig. 33

The steady state inductor current can be determined in either of two ways:

i) Use of Eq. 27, or

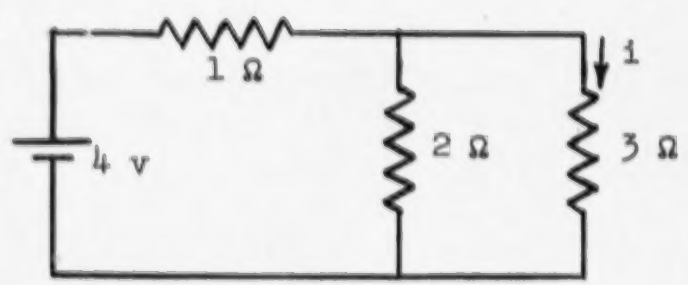
ii) Calculating $\underline{i}$ in Fig. 33, where the inductor has been replaced by a short circuit.

Determine the steady state value of $\underline{i}$ in each of the above two ways.

i) $\frac{22}{3}\, i = \frac{16}{3}$

$i = \frac{8}{11}$ ampere

ii)



$$i = \left(\frac{4}{1 + \frac{6}{5}} \right) \frac{2}{5} = \frac{8}{11}$$

$$\frac{1}{i - \frac{8}{11}} \frac{di}{dt} = -\frac{22}{3} \qquad (28)$$

$i(0+) = 1$

The reader should verify that Eq. 27 can be rewritten as Eq. 28. Integrating with respect to time on both sides of Eq. 28, as follows,

$$\int_0^t \left(\frac{1}{1 - \frac{8}{11}} \frac{di}{dt} \right) dt = \int_0^t \left(-\frac{22}{3} \right) dt \qquad (29)$$

will yield i(t) for $t \geq 0$. The left side of Eq. 29 is evaluated by making the change in integration variable already encountered.

Remember that i(0+) = 1 and carry out the required steps to get i(t), $t \geq 0$.

$$\int_0^t \frac{1}{1 - \frac{8}{11}} \frac{di}{dt}\, dt = \int_1^i \frac{di}{1 - \frac{8}{11}} = -\frac{22}{3} t$$

$$\ln\left(1 - \frac{8}{11}\right)\Big|_{i=1}^{i} = -\frac{22}{3} t$$

$$\ln \frac{11}{3}\left(1 - \frac{8}{11}\right) = -\frac{22}{3} t$$

$$i = \frac{8}{11} + \frac{3}{11} e^{-(22/3)t} \qquad (30)$$

To complete this problem, make a sketch of the function $i(t)$, $t \geq 0$, and state the value of the time constant.

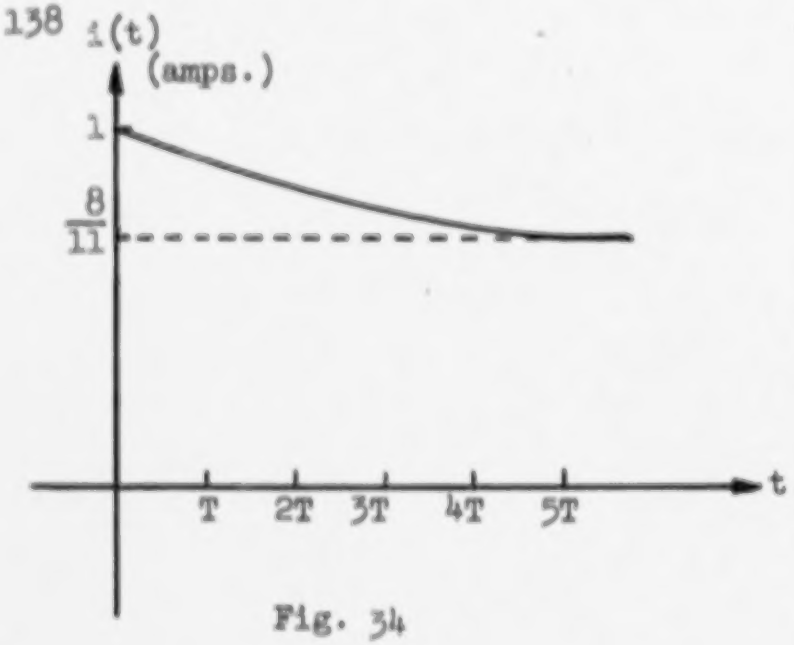


Fig. 34

$T = \frac{3}{22}$ seconds.

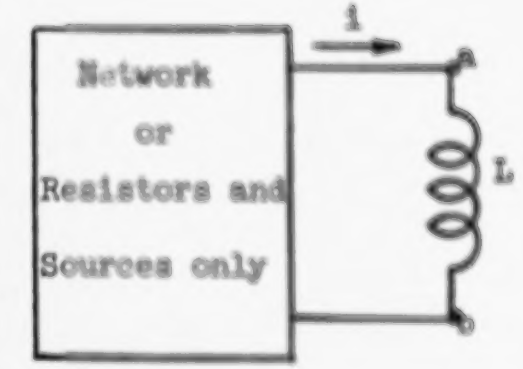


Fig. 35

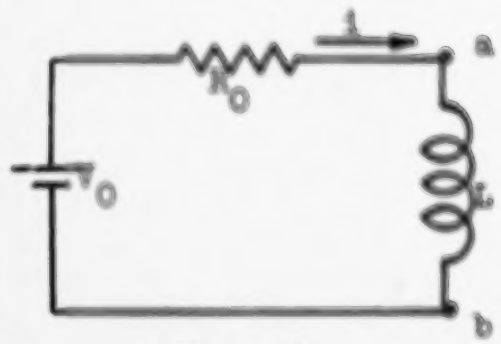


Fig. 36

There is a useful way of simplifying R-L network problems when there is only <u>one</u> inductor in the network. This case is depicted in Fig. 35, where the single inductor is drawn explicitly. The box contains only resistors and sources and can be replaced by its Thevenin equivalent. This results in the equivalent network of Fig. 36.

How is the time constant T related to R_0 in Fig. 36?

$$T = \frac{L}{R_O} \text{ seconds}$$

When the R-L network contains only one inductor and constant sources, the solution for the inductor current takes the form:

$$i(t) = a + be^{-(t/T)} \quad , \tag{31}$$

where a and b are constants.

Using the notation i_F for the final or steady state value of i(t), and i_0 for the initial value of i(t), $(i(0+) = i_0)$, solve for a and b in terms of i_0 and i_F.

$i_0 = i(0+) = a + b$

$i_F = i(\infty) = a$

Thus

$a = i_F$

$b = i_0 - i_F$

$$i(t) = i_F + (i_0 - i_F)e^{-t/T} \quad , \; t \geq 0 \qquad (32)$$

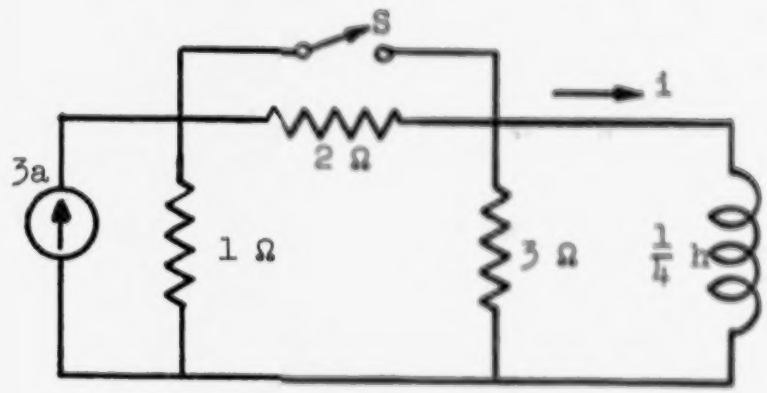


Fig. 37

When the three quantities, $i_0 = i(0+)$, i_p, and T are known, then Eq. 32 gives the complete solution. It is not necessary to write the equilibrium differential equation or to solve it.

To illustrate, we calculate $i(t)$, $t \geq 0$ in Fig. 37. The network is in the steady state at $t = 0$ when S is closed.

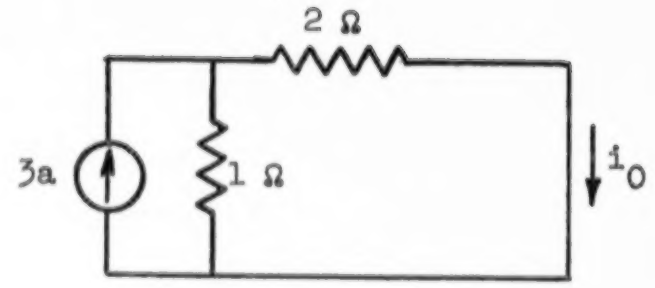
To start with, calculate:

a) $i(0+) = i_0$

b) i_F (the steady state current for $t \geq 0$)

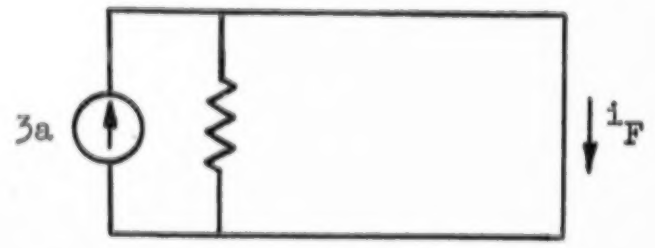
144

a)



$i_0 = 3 \frac{1}{3} = 1$ amp.

b)



$i_F = 3a$

)

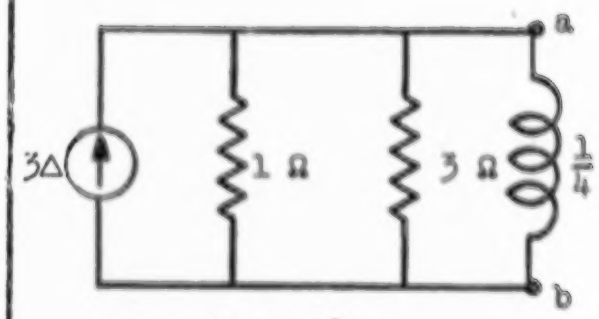
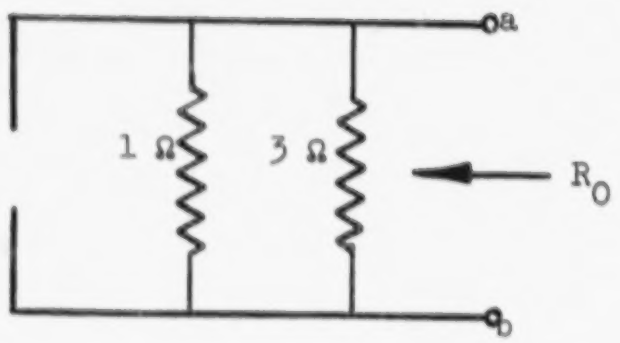


Fig. 38

For $t \geq 0$ the network of Fig. 37 becomes that of Fig. 38.

The time constant of the function i(t) can be calculated using the Thevenin equivalent of the network to the left of a-b in Fig. 38.

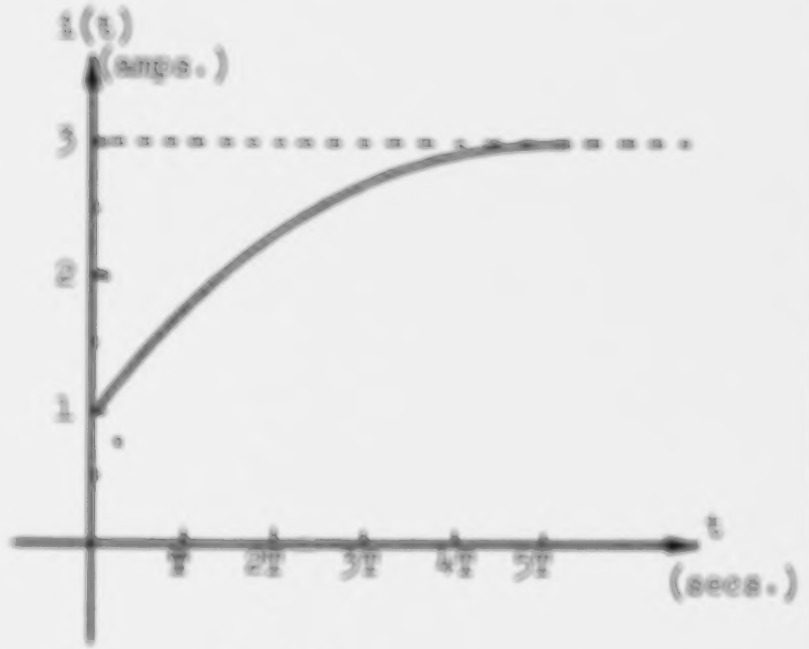
Determine the time constant, T.



$R_O = \frac{3}{4}$ ohm

$T = \frac{L}{R_O} = \frac{1}{3}$ second

Using Eq. 32, write down the explicit solution for $i(t)$, $t \geq 0$.
Sketch $i(t)$, $t \geq 0$.



$i(t) = 3 + (-2)e^{-3t}$

Summary

Complete the following statements:

1) When the network contains constant sources, resistors, and one inductor, the solution for any current or voltage is determined by the three quantities:
   a) _____ b) _____ c) _____.
2) The time constant is determined by replacing the rest of the network by _____.
3) The standard form of the solution for any current in this case can be written in terms of the quantities in 1) above as follows:

$$i(t) = _____$$

150

1) a) initial value
   b) final value
   c) time constant

2) ..... its Thevenin equivalent.

3) $i(t) = i_f + (i_0 - i_f)\, e^{-t/T}$